VIVEKANANDA COLLEGE THAKURPUKUR KOLKATA-700063

NAAC ACCREDITED 'A' GRADE

Topic: Fourier Transform

Course Title: signal and system

Paper: ELT-A-CC-4-10-TH

Unit: Fourier transform

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Name of the Department: Electronic science

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Fourier Transform :-

The Fourier transform is a mathematical technique that transfor a function of time, $x(t)$ to a function of frequency, $x(\omega)$

mathematical expression

$$F(J\omega) = \int_{-\alpha}^{+\alpha} f(t)\, e^{-j\omega t}\, dt$$

$$F^{-1}[F(j\omega)] = F(t)$$

$$F(j\omega) = \underbrace{|F(j\omega)|}_{\text{magnitude}} \cdot \underbrace{\phi(j\omega)}_{\text{phase}}.$$

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Functions :-

i) $f(t) = f_0 e^{-kt}$ for $t > 0$

$= 0$ for $t < 0$

$$f(J\omega) = f_0 \int_{0}^{+\infty} e^{-kt} e^{-j\omega t} \, dt$$

$$= f_0 \int_0^{\infty} e^{-(k+j\omega)t} \, dt$$

$$= \frac{f_0}{-(k+j\omega)} [0-1]$$

$$= \frac{f_0}{k+j\omega}$$

To find magnitude

$$|F(j\omega)| = \left[\frac{f_0}{(k+j\omega)} \cdot \frac{f_0}{(k-j\omega)} \right]^{1/2}$$

$$= \frac{f_0}{\sqrt{\omega^2 + k^2}}$$

$$\phi(j\omega) = \tan^{-1} y/x$$

$$= -\tan^{-1} \omega/k$$

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v) Derivative :-

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)\, e^{j\omega t}\, d\omega$$

$$\frac{d f(t)}{dt} = \frac{j\omega}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)\, e^{j\omega t}\, d\omega$$

$$\frac{d f(t)}{dt} = J\omega\, [f(t)]$$

$$F\left[\frac{d(f(t))}{dt}\right] = F[J\omega\, f(t)]$$

$$F\left[\frac{d f(t)}{dt}\right] = J\omega .\, F[f(t)]$$

$$F\left[\frac{d f(t)}{dt}\right] = J\omega .\, F(j\omega)$$

similarly,

$$F\left[\frac{d^n f(t)}{dt}\right] = (j\omega)^n .\, F(j\omega)$$

iii) Unit impulse function →

$$f(t) = \delta(t)$$

$$F[f(t)] = \int_{-\infty}^{+\infty} \delta(t) .\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} \delta(t - t_0)\, e^{-j\omega t}\, dt \quad \text{—(i}$$

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introduce scanning function →

$$\int_{-\infty}^{+\infty} f(t)\, \delta(t - t_0)\, dt = f(t_0) \quad \text{—(ii)}$$

Compare (i) and (ii)

$$f(t) = e^{-j\omega t}$$

let, $t_0 = 0$

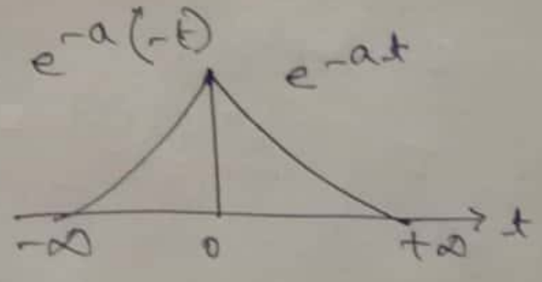
$$f(t_0) = e^{-j\omega t_0}\Big|_{t_0=0}$$

$$\boxed{f(t_0) = 1}$$

iv) <u>exponential function</u>

defined for both +ve (t) and -ve (t).

$$f(t) = e^{-a|t|}$$



$$F[f(t)] = \int_{-\infty}^{+\infty} e^{-a|t|}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{-a(-t)}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-a(t)}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

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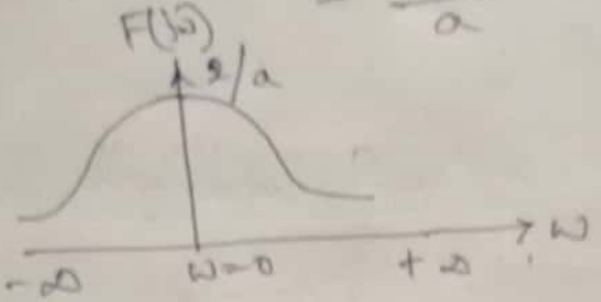
$$= \frac{1}{(a-j\omega)}[1-0] + \frac{1}{-(a+j\omega)}[0-1]$$

$$= \frac{1}{(a-j\omega)} + \frac{1}{(a+j\omega)}$$

$$F(j\omega) = \frac{2a}{a^2+\omega^2}$$

$$F|j\omega| = \frac{2a}{a^2}\Big|_{\omega=0}$$

$$= \frac{2}{a}$$



v) $f(t) = 1$

let, $\underset{a \to 0}{lt} \; e^{-a|t|} = 1$

$$F[f(t)] = \underset{a \to 0}{lt} \int_{-\infty}^{+\infty} e^{-a|t|} \cdot e^{-j\omega t} \, dt$$

$$= \underset{a \to 0}{lt} \; \frac{2a}{\omega^2+a^2} = 0 \; [\text{for finit } \omega]$$

limit has 0 value of all finite values of ω.

for $\omega = 0$ we use L'HOSPITAL RULE and differentiate the numerator and denominator with respect to a

$$\lim_{a \to 0} \frac{2a}{\omega^2 + a^2}$$

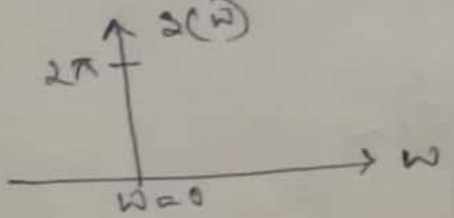
$$= \frac{2}{2a} = \frac{1}{a}$$

$$\lim_{a \to 0} F \frac{1}{a} = \infty$$

For $\omega = 0$ $\quad \lim_{a \to 0} \frac{2a}{\omega^2 + a^2} = \infty$

for any value of ω

$$\lim_{a \to 0} \frac{2a}{\omega^2 + a^2} = 0 \text{ or finite.}$$



strength of δ function →

$$\int_{-\infty}^{+\infty} \frac{2a}{\omega^2 + a^2} = 2\pi$$

ii) $f(t) = e^{j\omega_0 t}$

$F[f(t)] = 2\pi \, \delta(\omega - \omega_0)$ [displacement by ω_0]

[fourier transform of $1 = 2\pi \delta(\omega)$]

iii) <u>Cosine function</u> :-

$$F(\cos\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$= \frac{1}{2}\left[2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega + \omega_0)\right]$$

$$= \pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$

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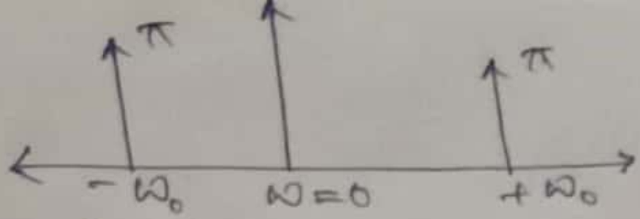
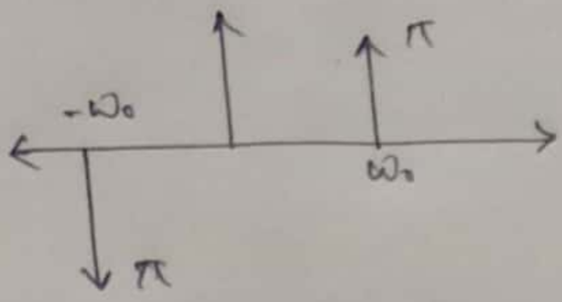


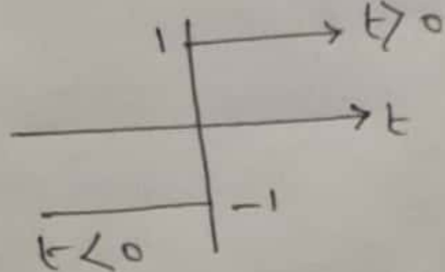
<u>sine function</u>

$$F(\sin\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$= -j\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$



viii) <u>Signum function</u>:-



$$\operatorname{sgn}(t) = 1 \quad t>0$$
$$= 0 \quad t=0$$
$$= -1 \quad t<0$$

$$F[\operatorname{sgn}(t)] = \lim_{a\to 0}\left[\int_{-\infty}^{+\infty} e^{-a|t|}\operatorname{sgn}(t)\, e^{-j\omega t}\, dt\right]$$

$$= \lim_{a\to 0}\left[\int_{-\infty}^{0} e^{-a(-t)}(-1)\, e^{-j\omega t}\, dt\right.$$
$$+ \lim_{a\to 0}\int_{0}^{+\infty} e^{-a(t)} \cdot 1\, e^{j\omega t}\, dt$$

$$= \lim_{a\to 0}\int_{0}^{+\infty} e^{-at}\, e^{-j\omega t}\, dt$$
$$- \lim_{a\to 0}\int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt$$

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$$= \underset{a\to 0}{lt} \int_{0}^{\infty} e^{-(j\omega+a)t}\,dt - \underset{a\to 0}{lt}\int_{-\infty}^{0} e^{-(j\omega-a)t}\,dt$$

$$= \underset{a\to 0}{lt} \left[\frac{e^{-(j\omega+a)t}}{-(j\omega+a)}\right]_0^{\infty} - \underset{a\to 0}{lt} \left.\frac{e^{-(j\omega-a)t}}{-(j\omega-a)}\right|_{-\infty}^{0}$$

$$= \underset{a\to 0}{lt} \frac{1}{-(j\omega+a)}[0-1] - \underset{a\to 0}{lt} \frac{1}{-(j\omega-a)}[1-0]$$

$$= \underset{a\to 0}{lt} \frac{1}{a+j\omega} + \frac{1}{j\omega-a}$$

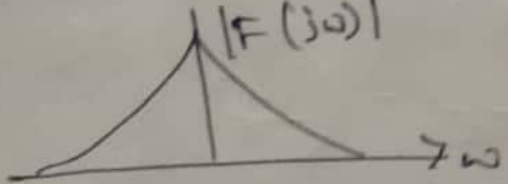
$$= \frac{1}{j\omega} + \frac{1}{j\omega}$$

$$= \frac{2}{j\omega}$$

$$|F(j\omega)| = \left[\left(\frac{2}{j\omega}\right)\left(\frac{2}{-j\omega}\right)\right]^{1/2}$$

$$= \frac{2}{\omega}$$

$$\phi(j\omega) = -\tan^{-1}\frac{\omega}{0} = -90^\circ$$



another function :-

$$f(t) = \frac{1}{2}[1 + \operatorname{sgn}(t)]$$

$$F(j\omega) = \frac{1}{2}\left[2\pi\delta(\omega) + \frac{2}{j\omega}\right]$$

so, $F[f(t)] = L[f(t)]\Big|_{s=j\omega}$

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Operational transform

i) Fourier transform of $k f(t)$

$$F[k f(t)] = k F(j\omega)$$

ii) Addition | subtraction :-

$$F[f_1(t) - f_2(t) + f_3(t)]$$
$$= F_1(j\omega) - F_2(j\omega) + F_3(j\omega)$$

iii) Differentiation

$$F\left[\frac{d^n f(t)}{dt^n}\right] = (j\omega)^n \cdot F(j\omega)$$

iv) integration

$$F[f(t)] = \frac{F(j\omega)}{j\omega}$$

v) scale changer :-

$$F[f(at)] = \frac{1}{a} F\left(\frac{j\omega}{a}\right)$$

vi) Translation in domain of time.

$$F[f(t-a)] = e^{-j\omega a} F(j\omega)$$

vii) Translation in frequency domain

$$F[e^{j\omega_0 t} f(t)] = F[j(\omega-\omega_0)]$$

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viii) Modulation

$$F[f(t)\cdot \cos\omega_0 t] = \frac{1}{2} F[j(\omega-\omega_0)] + \frac{1}{2} F[j(\omega+\omega_0)]$$

ix) a) Convolation in time domain gives multiplication in frequency domain

$$h(t) = \int_{-\infty}^{+\infty} x(\lambda)\cdot y(t-\lambda)\, d\lambda.$$

$$F[h(t)] = H(j\omega) = X(j\omega)\cdot Y(j\omega)$$

b) Convolation in frequency domain is given by fourier transform of the product of two time function

$$f(t) = f_1(t)\cdot f_2(t)$$

$$F[f(t)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_1(j\lambda)\cdot F_2\{j(\omega-\lambda)\}\, d\lambda$$

application :-

i) Thus if $x(t)$ is input excitation and $y(t)$ is impulse response of the circuit and the output response is given by

$$h(t) = \int_{-\infty}^{+\infty} x(\tau)\cdot y(t-\tau)\, d\tau$$

$x(t)$ = input , $y(t)\rightarrow$ impulse response

$h(t)$ = output

$H(j\omega) = X(j\omega)\cdot Y(j\omega)$

$\therefore Y(j\omega) = \frac{H(j\omega)}{X(j\omega)}$ [transfer function]

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