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RECTANGULAR (OR UNIFORM) DISTRIBUTION

Definition. *A random variable X is said to have a continuous rectangular (uniform) distribution over an interval* (a, b), i.e., $(-\infty < a < b < \infty)$, *if its p.d.f. is given by :*

$$f(x\,;a,b) = \begin{cases} \dfrac{1}{b-a}, & \text{if } a < x < b \\ 0, & \text{otherwise} \end{cases} \qquad \ldots (9{\cdot}19)$$

Remarks 1. a and b, $(a < b)$ are the two parameters of the distribution. The distributin is called uniform distribution on (a, b) since it assumes a constant (uniform) value for all x in (a, b).

2. The distribution is also known as rectangular distribution, since the curve $y = f(x)$ describes a rectangle over the x-axis and between the ordinates at $x = a$ and $x = b$.

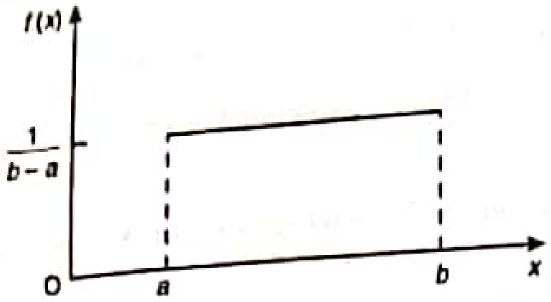
3. A uniform or rectangular variate X on the interval (a, b) is written as : $X \sim U[a, b]$ or $X \sim R[a, b]$.

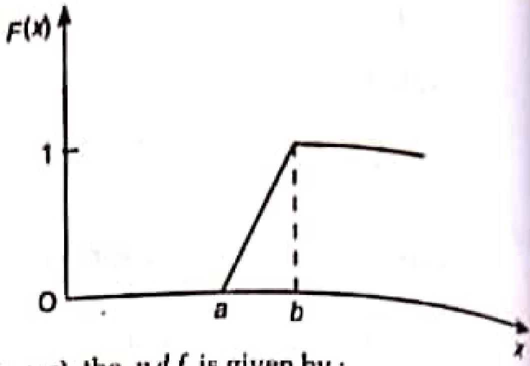
4. The cumulative distribution function F(x) is given by :

$$F(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a < x < b \\ 1, & x \ge b \end{cases} \qquad \ldots (9{\cdot}19a)$$

Since $F(x)$ is not continuous at $x = a$ and $x = b$, it is not differentiable at these points. Thus $\dfrac{d}{dx}F(x) = f(x) = \dfrac{1}{b-a} \neq 0$, exists everywhere except at the points $x = a$ and $x = b$ and consequently p.d.f. $f(x)$ is given by (9·19).

5. The graphs of uniform *p.d.f.* $f(x)$ and the corresponding distribution function $F(x)$ are given below.





6. For a rectangular or uniform variate X in $(-a, a)$, the *p.d.f.* is given by :

$$f(x) = \begin{cases} \frac{1}{2a}, & -a < x < a \\ 0, & \text{otherwise} \end{cases} \qquad \dots (9\cdot19b)$$

9·3·1. Moments of Rectangular Distribution. Let $X \sim U[a, b]$.

$$\mu_r' = \int_a^b x^r f(x)\,dx = \frac{1}{b-a}\int_a^b x^r\,dx = \frac{1}{b-a}\left(\frac{b^{r+1}-a^{r+1}}{r+1}\right) \qquad \dots (9\cdot20)$$

In particular

$$\text{Mean} = \mu_1' = \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right) = \frac{b+a}{2} \qquad \dots (9\cdot20a)$$

and
$$\mu_2' = \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right) = \frac{1}{3}(b^2+ab+a^2)$$

$$\therefore \quad \text{Variance} = \mu_2' - \mu_1'^2 = \frac{1}{3}(b^2+ab+a^2) - \left\{\frac{1}{2}(b+a)\right\}^2 = \frac{1}{12}(b-a)^2 \qquad \dots (9\cdot20b)$$

9·3·2. M.G.F. of Rectangular Distribution is given by :

$$M_X(t) = \int_a^b e^{tx} f(x)\,dx = \int_a^b \frac{e^{tx}}{b-a}\,dx = \frac{e^{bt}-e^{at}}{t\,(b-a)}, \; t \neq 0 \qquad \dots (9\cdot20c)$$

9·3·3. Characteristic Function of Rectangular Distribution is given by :

$$\phi_X(t) = \int_a^b e^{itx}\,dx = \frac{e^{ibt}-e^{iat}}{it\,(b-a)}, \; t \neq 0. \qquad \dots (9\cdot20d)$$

9·3·4. Mean Deviation about Mean, η of Rectangular Distribution is given by :

$$\eta = E\left|X - \text{Mean}\right| = \int_a^b \left|x - \text{Mean}\right| f(x)\,dx$$

$$= \frac{1}{b-a}\int_a^b \left|x - \frac{a+b}{2}\right| dx = \frac{1}{b-a}\int_{-(b-a)/2}^{(b-a)/2} |t|\,dt, \text{ where } t = x - \frac{a+b}{2}$$

$$= \frac{1}{b-a}\cdot 2\int_0^{(b-a)/2} t\,dt = \frac{b-a}{4} \qquad \dots (9\cdot20e)$$

$$\frac{(n+1)!}{k!(n+1-k)!}\cdot x^k\cdot\frac{(n+1-k)!}{(n-k)!}\cdot(1-x)^{n-k}\,dx=\binom{n}{k}(n+1)\,x^k(1-x)^{n-k}\,dx$$

Verification. $I=\int_0^1\binom{n}{k}(n+1)\,x^k(1-x)^{n-k}\,dx=\binom{n}{k}(n+1)\,\beta(k+1,n-k+1)$

on using beta-Integral :

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx=B(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)};\ m>0,n>0.$$

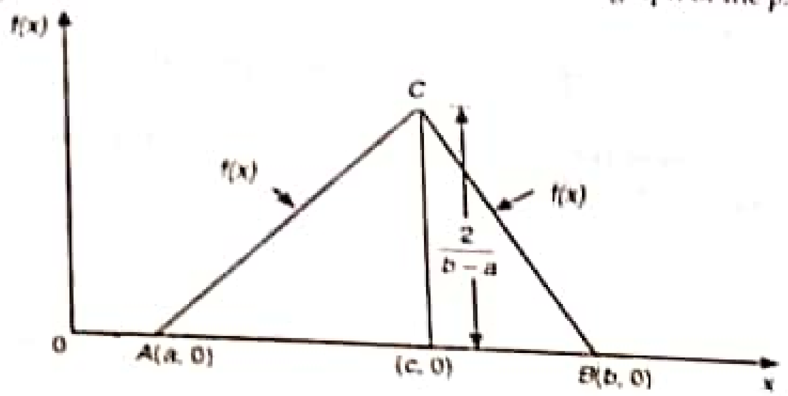
$$I=\frac{n!(n+1)}{k!(n-k)!}\cdot\frac{k!(n-k)!}{(n+1)!}=1.$$

TRIANGULAR DISTRIBUTION

Definition. *A random variable X is said to have a triangular distribution in the interval (a, b), if its p.d.f. is given by :*

$$f(x)=\begin{cases}\dfrac{2(x-a)}{(b-a)(c-a)}, & a<x\le c\\[2mm] \dfrac{2(b-x)}{(b-a)(b-c)}, & c<x<b\end{cases}\qquad\ldots(9\cdot21)$$

Remarks 1. We write X ~ Trg. (a, b), with peak at $x=c$. The graph of the *p.d.f.* is shown in following diagram :



2. The distribution is so called because the graph of its *p.d.f.* is a triangle with peak at $x=c$.

3. The *m.g.f.* of Trg (a, b) variate, with peak at $x=c$ is given by :

$$M_X(t)=\int_a^b e^{tx}f(x)\,dx=\left(\int_a^c+\int_c^b\right)e^{tx}f(x)\,dx$$

$$=\frac{2}{(b-a)(c-a)}\int_a^c e^{tx}(x-a)\,dx+\frac{2}{(b-a)(b-c)}\int_c^b e^{tx}(b-x)\,dx$$

$$=\frac{2}{t^2}\left\{\frac{e^{at}}{(a-b)(a-c)}+\frac{e^{ct}}{(c-a)(c-b)}+\frac{e^{bt}}{(b-a)(b-c)}\right\},\ a<c<b\qquad\ldots(9\cdot21a)$$

(On integration by parts.)

4. In particular, taking $a=0$, $c=1$ and $b=2$, in (9·21), the *p.d.f.* of the Trg (0, 2) variate with peak at $x=1$ is given by :

$$f(x)=\begin{cases}x; & 0\le x\le 1\\ 2-x; & 1\le x\le 2\\ 0, & \text{otherwise}\end{cases}\qquad\ldots(9\cdot21b)$$

and its *m.g.f.* is : $M_X(t) = (e^t - 1)^2 / t^2$,

which is left as an exercise to the reader. ... (9·21*c*)

5. In particular, replacing a by $-2a$, b by $2a$ and c by 0, the *p.d.f.* of triangular distribution on the interval $(-2a, 2a)$ with peak at $x = 0$ is given by :

$$f(x) = \begin{cases} (2a + x)/4a^2 ; & -2a < x < 0 \\ (2a - x)/4a^2 ; & 0 < x < 2a \end{cases} \quad \text{... (9·21}d\text{)}$$

The *m.g.f.* of (9·21*d*) is given by :

$$M_X(t) = \int_{-2a}^{2a} e^{tx} f(x)\, dx = \frac{1}{4a^2} \left\{ \int_{-2a}^{0} e^{tx} (2a + x)\, dx + \int_{0}^{2a} e^{tx} (2a - x)\, dx \right\}$$

$$= \frac{1}{4a^2} \left[e^{tx} \left\{ \frac{2a + x}{t} - \frac{1}{t^2} \right\} \right]_{-2a}^{0} + \frac{1}{4a^2} \left[e^{tx} \left\{ \frac{2a - x}{t} + \frac{1}{t^2} \right\} \right]_{0}^{2a} \quad \text{[On integrating by parts]}$$

$$= \frac{1}{4a^2} \left[-\frac{2}{t^2} + \frac{1}{t^2} \left\{ e^{2at} + e^{-2at} \right\} \right] = \frac{1}{4a^2 t^2} \left\{ e^{2at} + e^{-2at} - 2 \right\} = \left\{ \frac{1}{2at} (e^{at} - e^{-at}) \right\}^2$$

Aliter. We may obtain (9·21) directly from (9·2*a*) on replacing a by $-2a$, b by $2a$ and c by 0.

Example 9·29. *If X and Y are i.i.d. U* $[-a, a]$ *variates, find the p.d.f. of* $Z = X + Y$ *and identify the distribution.*

Solution. Since X and Y are *i.i.d.* $U\ [-a, a]$, we have : [*c.f.* § 9·3·2],

$$M_X(t) = M_Y(t) = (e^{at} - e^{-at})/(2\,at) \quad \text{...(*)}$$

$$M_{X+Y}(t) = M_X(t)\, M_Y(t) = \left\{ \frac{1}{2at} (e^{at} - e^{-at}) \right\}^2, \text{ since } X \text{ and } Y \text{ are independent.}$$

But, this is the *m.g.f.* of Trg $(-2a, 2a)$ variate with peak at $x = 0$.

[*c.f.* Remark 5, equation (9·21*e*)]

Hence by uniqueness theorem of *m.g.f.*, $Z = X + Y \sim$ Trg $(-2a, 2a)$ with *p.d.f.* as given in (9·21*d*), Remark 5.

Aliter.

$$M_{X+Y}(t) = \frac{1}{4a^2t^2} (e^{2at} - 2 + e^{-2at}) \quad \text{[From (**)]}$$

$$= \frac{2}{t^2} \left[\frac{e^{-2at}}{(-2a - 0)(-2a - 2a)} + \frac{e^{0 \cdot t}}{(0 + 2a)(0 - 2a)} + \frac{e^{2at}}{(2a - 0)(2a + 2a)} \right]$$

which is of the form (9·21*a*), [*c.f.* Remark 3], with a replaced by $-2a$ and b replaced by $2a$ and c by 0. Hence $X + Y \sim$ Trg $(-2a, 2a)$ with peak at $x = 0$ and *p.d.f.* $p(x)$ given in (9·21*d*).

Remarks 1. The distribution of $X + Y$ has also been obtained in *Example* 9·27.

2. Similarly we can find the distribution of $X - Y$.

$$M_{X-Y}(t) = M_X(t) . M_Y(-t) = \left\{ \frac{1}{2at} (e^{at} - e^{-at}) \right\}^2 \quad \text{[From (*)]}$$

$\Rightarrow$ $X - Y \sim$ Trg $(-2a, 2a)$, with peak at $x = 0$.

9·5. GAMMA DISTRIBUTION

Definition. *A r.v. X is said to have a gamma distribution with parameter* $\lambda > 0$, *if its p.d.f. is given by :*

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\lambda - 1}}{\Gamma(\lambda)} ; & \lambda > 0,\ 0 < x < \infty \\ 0, & \text{otherwise} \end{cases} \quad \text{...(9·22)}$$

Remarks 1. X is known as a Gamma variate with parameter λ and referred to as a $\gamma(\lambda)$ variate.

2. The function $f(x)$ defined above represents a probability function, since

$$\int_0^\infty f(x)\,dx = \frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-x}x^{\lambda-1}\,dx = \frac{1}{\Gamma(\lambda)}\cdot\Gamma(\lambda) = 1$$

3. A continuous random variable X having the following *p.d.f.* is said to have a gamma distribution with two parameters a and λ.

$$f(x) = \begin{cases} \dfrac{a^\lambda}{\Gamma(\lambda)}e^{-ax}x^{\lambda-1}; & a > 0, \lambda > 0, 0 < x < \infty \\ 0, & \text{otherwise} \end{cases} \qquad \ldots(9.22a)$$

Here $X \sim \gamma(a, \lambda)$. Taking $a = 1$ in (9·22a), we get (9·22). Hence we may write $X \sim \gamma(\lambda) = \gamma(1, \lambda)$.

4. The *cumulative distribution function*, called incomplete gamma function is defined as :

$$F_X(x) = \begin{cases} \displaystyle\int_0^x f(u)\,du = \frac{1}{\Gamma(\lambda)}\int_0^x e^{-u}u^{\lambda-1}\,du, & x > 0 \\ 0, & \text{otherwise} \end{cases} \qquad \ldots(9.22b)$$

9·5·1. M.G.F. of Gamma Distribution. M.G.F. about origin is given by :

$$M_x(t) = E(e^{tX}) = \int_0^\infty e^{tx}f(x)\,dx = \frac{1}{\Gamma(\lambda)}\int_0^\infty e^{tx}e^{-x}x^{\lambda-1}\,dx$$

$$= \frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-(1-t)x}x^{\lambda-1}\,dx = \frac{1}{\Gamma(\lambda)}\cdot\frac{\Gamma(\lambda)}{(1-t)^\lambda},\ |t| < 1$$

$$\therefore \quad M_X(t) = (1-t)^{-\lambda},\ |t| < 1 \qquad \ldots(9.23)$$

9·5·2. Cumulant Generating Function of Gamma Distribution. The cumulant generating function $K_X(t)$ is given by :

$$K_X(t) = \log M_X(t) = \log(1-t)^{-\lambda} = -\lambda\log(1-t);\ |t| < 1$$

$$= \lambda\left(t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \ldots\right)$$

$\therefore$ Mean $= \kappa_1 =$ Coefficient of t in $K_X(t) = \lambda$

Variance $= \mu_2 = \kappa_2 =$ Coefficient of $\frac{t^2}{2!}$ in $K_X(t) = \lambda$

Hence if $X \sim \gamma(\lambda)$, Mean = Variance = λ.

$\mu_3 = \kappa_3 =$ Coefficient of $\frac{t^3}{3!}$ in $K_X(t) = 2\lambda$

$\kappa_4 =$ Coefficient of $\frac{t^4}{4!}$ in $K_X(t) = 6\lambda \Rightarrow \mu_4 = \kappa_4 + 3\kappa_2^2 = 6\lambda + 3\lambda^2$.

Hence $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{4\lambda^2}{\lambda^3} = \dfrac{4}{\lambda}$ and $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = 3 + \dfrac{6}{\lambda}$.

Remarks 1. Like Poisson distribution, the mean and variance of the Gamma distribution are also equal. However, Poisson distribution is discrete while Gamma distribution is continuous.

2. Limiting form of Gamma distribution as $\lambda \to \infty$. We know that if $X \sim \gamma(\lambda)$, then $E(X) = \lambda = \mu$, (say) and Var $(X) = \lambda = \sigma^2$, (say). Then standard gamma variate is given by : $Z = \dfrac{X-\mu}{\sigma} = \dfrac{X-\lambda}{\sqrt{\lambda}}$

$$M_Z(t) = \exp(-\mu t/\sigma) M_X(t/\sigma) = \exp(-\mu t/\sigma)\left(1-\frac{t}{\sigma}\right)^{-\lambda} = e^{-t\lambda/\sqrt{\lambda}}\left(1-\frac{t}{\sqrt{\lambda}}\right)^{-\lambda}$$

$$\Rightarrow \quad K_Z(t) = \sqrt{\lambda}\,.\,t - \lambda \log\left(1-\frac{t}{\sqrt{\lambda}}\right) = -\sqrt{\lambda}\ t - \lambda\left(\frac{t}{\sqrt{\lambda}} + \frac{t^2}{2\lambda} + \frac{t^3}{3\lambda^{3/2}} + \ldots\right)$$

$$= -\sqrt{\lambda}\ t + \sqrt{\lambda}\ t + \frac{t^2}{2} + 0(\lambda^{-1/2}),$$

where $0(\lambda^{-1/2})$ are terms containing $\lambda^{1/2}$ and higher powers of λ in the denominator.

$$\therefore \quad \lim_{\lambda\to\infty} K_Z(t) = \frac{t^2}{2} \quad \Rightarrow \quad \lim_{\lambda\to\infty} M_Z(t) = \exp(t^2/2)$$

which is the *m.g.f.* of a Standard Normal Variate. Hence by uniqueness theorem of *m.g.f.*, Standard Gamma variate tends to Standard Normal Variate as $\lambda \to \infty$. In other words, *Gamma distribution tends to Normal distribution for large value of parameter* λ.

3. For the two parameter gamma distribution (9·22 *a*), we have

$$M_X(t) = \left(1-\frac{t}{a}\right)^{-\lambda};\ t < a. \qquad \ldots(9\cdot23\ a)$$

Proof is left as an excercise to the reader.

Also $\quad K_X(t) = -\lambda \log\left(1-\frac{t}{a}\right) = \lambda\left\{\frac{t}{a} + \frac{1}{2}\left(\frac{t}{a}\right)^2 + \frac{1}{3}\left(\frac{t}{a}\right)^3 + \ldots\right\};\ t < a$

$\therefore \quad$ Mean $= \kappa_1 = \lambda/a$ and Variance $= \kappa_2 = \lambda/a^2 =$ Mean$/a$ $\qquad \ldots(9\cdot23\ b)$

Hence Variance > Mean if $a < 1$; Variance = Mean if $a = 1$; and Variance < Mean if $a > 1$.

9·5·3. Additive Property of Gamma Distribution. *The sum of independent Gamma variates is also a Gamma variate. More precisely, if* $X_1, X_2, \ldots, X_k$ *are independent Gamma variates with parameters* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *respectively then* $X_1 + X_2 + \ldots + X_k$ *is also a Gamma variate with parameter* $\lambda_1 + \lambda_2 + \ldots + \lambda_k$.

Proof. Since X_i is a $\gamma(\lambda_i)$ variate, $M_{X_i}(t) = (1-t)^{-\lambda_i}$

The *m.g.f.* of the sum $X_1 + X_2 + \ldots + X_k$ is given by :

$$M_{X_1+X_2+\ldots X_k}(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_k}(t), \qquad (\because X_1, X_2, \ldots, X_k \text{ are independent.})$$

$$= (1-t)^{-\lambda_1}(1-t)^{-\lambda_2}\ldots(1-t)^{-\lambda_k} = (1-t)^{-(\lambda_1+\lambda_2+\ldots+\lambda_k)}$$

which is the *m.g.f.* of a Gamma variate with parameter $\lambda_1 + \lambda_2 + \ldots + \lambda_k$. Hence the result follows by the uniqueness theorem of *m.g.f.*'s.

Remark. In general, if $X_i \sim \gamma(a, \lambda_i),\ i = 1, 2, \ldots, n$ are independent *r.v.*' s, then

$$\sum_{i=1}^{n} X_i \sim \gamma\left(a,\ \sum_{i=1}^{n} \lambda_i\right).$$

Solution. (i) $\mu_r' = \int_{-\infty}^{\infty} x^r f(x)\,dx = \int_{-\infty}^{\infty} x^r \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$

$$\therefore \quad \frac{d\mu_r'}{d\sigma} = \int_{-\infty}^{\infty} \frac{x^r}{\sqrt{2\pi}}\left(-\frac{1}{\sigma^2}\right)\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]dx + \int_{-\infty}^{\infty} \frac{x^r}{\sqrt{2\pi}\,\sigma}\left\{-\frac{(x-\mu)^2}{2}\times\left(\frac{-2}{\sigma^3}\right)\right\}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]dx$$

$$= -\frac{1}{\sigma}\int_{-\infty}^{\infty} x^r . f(x)\,dx + \frac{1}{\sigma^3}\int_{-\infty}^{\infty} x^r (x-\mu)^2 . f(x)\,dx$$

$$= -\frac{1}{\sigma}\cdot\mu_r' + \frac{1}{\sigma^3}\int_{-\infty}^{\infty} (x^{r+2} - 2\mu x^{r+1} + \mu^2 x^r) f(x)\,dx$$

$$= -\frac{1}{\sigma}\cdot\mu_r' + \frac{1}{\sigma^3}\left(\mu'_{r+2} - 2\mu\,\mu'_{r+1} + \mu^2\mu_r'\right)$$

$$\Rightarrow \quad \sigma^3\frac{d\mu_r'}{d\sigma} = -\sigma^2\mu'_r + \mu'_{r+2} - 2\mu . \mu'_{r+1} + \mu^2\mu'_r$$

$$\Rightarrow \quad \mu'_{r+2} = 2\mu . \mu'_{r+1} + (\sigma^2 - \mu^2)\mu'_r + \sigma^3 . \frac{d\mu_r'}{d\sigma}.$$

(ii) $\mu_{2r} = \int_{-\infty}^{\infty} (x-\mu)^{2r} . f(x)dx = \int_{-\infty}^{\infty} (x-\mu)^{2r}\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]dx$

$$\therefore \quad \frac{d\mu_{2r}}{d\sigma} = \int_{-\infty}^{\infty}\frac{(x-\mu)^{2r}}{\sqrt{2\pi}}\left(-\frac{1}{\sigma^2}\right).\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]dx + \int_{-\infty}^{\infty}\frac{(x-\mu)^{2r}}{\sqrt{2\pi}.\sigma}\cdot\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right].\left\{-\frac{(x-\mu)^2}{2}\left(\frac{-2}{\sigma^3}\right)\right\}dx$$

$$= -\frac{1}{\sigma}\int_{-\infty}^{\infty}(x-\mu)^{2r} f(x)\,dx + \frac{1}{\sigma^3}\int_{-\infty}^{\infty}(x-\mu)^{2r+2} . f(x)dx$$

$$= -\frac{1}{\sigma}\mu_{2r} + \frac{1}{\sigma^3}\mu_{2r+2}$$

$$\Rightarrow \quad \mu_{2r+2} = \sigma^2\mu_{2r} + \sigma^3\frac{d\mu_{2r}}{d\sigma}.$$

9·2·15. Log-normal Distribution. The positive *r.v.* X is said to have a log-normal distribution if $\log_e X$ is normally distributed.

Let $Y = \log_e X \sim N(\mu, \sigma^2)$. For $x > 0$,

$F_X(x) = P(X \le x) = P(\log_e X \le \log_e x) = P(Y \le \log_e x)$

(Since $\log X$ is monotonic increasing function.)

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\log x}\exp\{-(y-\mu)^2/2\sigma^2\}\,dy \qquad [\text{Since } Y \sim N(\mu, \sigma^2)]$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_0^x \exp\{-(\log u - \mu)^2/2\sigma^2\}\frac{du}{u}, \qquad (y = \log u)$$

For $x \le 0$, $F_X(x) = P(X \le x) = 0$, because X is a positive *r.v.*

Let us define

$$f_X(u) = \begin{cases} \frac{1}{u\sigma\sqrt{2\pi}} \cdot \exp\{-(\log u-\mu)^2/2\sigma^2\}, u>0 \\ 0, u \le 0 \end{cases} \quad \dots (9\cdot17)$$

Then $F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$, for every x and hence $f(x)$ defined in (9·17) is a *p.d.f.*

Remark. If $X \sim N(\mu, \sigma^2)$, then $Y = e^X$, is called a log-normal random variable, since its ...rithm $\log Y = X$, is a normal *r.v.*

Moments. The rth moment about origin is given by :

$$\mu_r' = E(X^r) = E(e^{rY}) \quad [\because Y = \log X \Rightarrow X = e^Y]$$

$$= M_Y(r) \quad (\text{m.g.f. of } Y, r \text{ being the parameter})$$

$$= \exp(\mu r + \tfrac{1}{2} r^2 \sigma^2) \quad [\because Y \sim N(\mu, \sigma^2)] \quad \dots (9\cdot18)$$

Remarks 1. In particular if we take $\mu = \log \alpha, \alpha > 0$, *i.e.*, $\log X \sim N(\log \alpha, \sigma^2)$, then

$$\mu_r' = E(X^r) = \exp\{r.\log\alpha + \tfrac{1}{2} r^2\sigma^2\} = \alpha^r.\exp\{r^2\sigma^2/2\} \quad \dots (9\cdot18a)$$

$\therefore$ Mean $= \mu_1' = \alpha e^{\sigma^2/2}$ and $\mu_2 = \mu_2' - \mu_1'^2 = \alpha^2 e^{\sigma^2}(e^{\sigma^2} - 1)$

2. Log normal distribution arises in problems of economics, biology, geology, and ...ability theory. In particular, it arises in the study of dimensions of particles under ...verisation.

3. If $X_1, X_2, \dots, X_n$ is a set of independently identically distributed random variables such ...at mean of each $\log X_i$ is μ and its variance is σ^2, then the product $X_1 . X_2 \dots . X_n$ is ...mptotically distributed according to logarithmic normal distribution and with mean μ and ...riance $n\sigma^2$.

...RECTANGULAR (OR UNIFORM) DISTRIBUTION