VIVEKANANDA COLLEGE
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NAAC ACCREDITED 'A' GRADE

Topic: Geometric_Negetive_Binomial Distribution

Course Title: Discrete Probability Distribution

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Geometric Distribution (1)

Consider a series of independent Bernoullian trials.
Let X be a random variable denoting the no. of trials required to get the 1st success. The probability distribution is defined as

$$P[X=x] = P\{x \text{ trials a required to get the 1st success}\}$$
$$= P\{F \cap F \cap \cdots \cap F \cap S\}$$
$$= q^{x-1} p$$

Hence the r.v X is said to follow a geometric distribution of 1st kind with parameter p iff its p.m.f is given by

$$f(x) = q^{x-1} p \quad ; \quad x = 1, 2, \ldots$$
$$= 0 \quad ; \quad \text{o.w.}$$

Expectation :-

$$E(X) = \sum_{x=1}^{\infty} x \, f(x)$$
$$= \sum_{x=1}^{\infty} x \cdot q^{x-1} p$$
$$= p \sum_{x=1}^{\infty} \frac{d}{dq}(q^x)$$
$$= p \frac{d}{dq}\left[\sum_{x=1}^{\infty} q^x\right] = p \frac{d}{dq}\left[\frac{q}{1-q}\right] = p \frac{1-q+q}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

Variance :-

$$V(X) = E(X^2) - E^2(X)$$

Now
$$E(X^2) = \sum_{x=1}^{\infty} x^2 f(x)$$
$$= \sum_{x=1}^{\infty} x(x-1) f(x) + \sum_{x=1}^{\infty} x f(x)$$
$$= \sum_{x=1}^{\infty} x(x-1) q^{x-1} p + E(X)$$
$$= pq \sum_{x=1}^{\infty} x(x-1) q^{x-2} + \frac{1}{p}$$
$$= pq \sum_{x=1}^{\infty} \frac{d^2}{dq^2}[q^x] + \frac{1}{p}$$
$$= pq \frac{d^2}{dq^2}\left[\sum_{x=1}^{\infty} q^x\right] + \frac{1}{p}$$
$$= pq \frac{d^2}{dq^2}\left[\frac{q}{1-q}\right] + \frac{1}{p}$$
$$= pq \frac{d}{dq}\left[\frac{1}{(1-q)^2}\right] + \frac{1}{p}$$
$$= pq \frac{2}{(1-q)^3} + \frac{1}{p} = \frac{2pq}{p^3} + \frac{1}{p} = \frac{2q+p}{p^2}$$

$$\therefore V(X) = \frac{2q+p}{p^2} - \frac{1}{p^2} = \frac{q+(q+p)-1}{p^2} = \frac{q}{p^2}$$

Moment Generating Function (MGF)

$$M_x(t) = E(e^{xt})$$
$$= \sum_{x=1}^{\infty} e^{xt} q^{x-1} p$$
$$= pe^t \sum_{x=1}^{\infty} (qe^t)^{x-1} = pe^t \frac{1}{1-qe^t} \qquad [\because qe^t < 1]$$
$$= \frac{pe^t}{1-qe^t}$$

$$\frac{d}{dt} M_x(t)\Big|_{t=0} = \frac{(1-qe^t)pe^t - pe^t(-qe^t)}{(1-qe^t)^2}\Big|_{t=0} = \frac{pe^t}{(1-qe^t)^2}\Big|_{t=0} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$
$$= E(x)$$

$$\frac{d^2}{dt^2} M_x(t)\Big|_{t=0} = \frac{d}{dt}\left[\frac{pe^t}{(1-qe^t)^2}\right]_{t=0}$$
$$= \frac{(1-qe^t)^2 pe^t - pe^t\, 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}\Big|_{t=0}$$
$$= \left[\frac{pe^t}{(1-qe^t)^2} + \frac{2pe^t qe^t}{(1-qe^t)^3}\right]\Big|_{t=0}$$
$$= \frac{p}{p^2} + \frac{2pq}{p^3} = \frac{1}{p} + \frac{2q}{p^2} = E(x^2)$$

$$\therefore V(x) = \frac{1}{p} + \frac{2q}{p^2} - \frac{1}{p^2} = \frac{p+q+q-1}{p^2} = \frac{q}{p^2}$$

Probability Generating Function [PGF]

$$P_x(t) = E(t^x)$$
$$= \sum_{x=1}^{\infty} t^x f(x)$$
$$= \sum_{x=1}^{\infty} t^x q^{x-1} p = pt \sum_{x=1}^{\infty} (tq)^{x-1} = \frac{pt}{1-tq}, \quad (tq) < 1$$

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Recursion Relation :-

$$\mu_r = E\left(x - \frac{1}{p}\right)^r \quad , \text{ here, } \mu = \frac{1}{p}$$

$$= \sum_{x=1}^{\infty} \left(x - \frac{1}{p}\right)^r p\,(1-p)^{x-1}$$

$$\therefore \frac{d}{dp}\mu_r = r\sum_{x=1}^{\infty}\left(x-\frac{1}{p}\right)^{r-1}\frac{1}{p^2}\cdot p\,(1-p)^{x-1} + \sum_{x=1}^{\infty}\left(x-\frac{1}{p}\right)^r (1-p)^{x-1}$$

$$+ \sum_{x=1}^{\infty}\left(x-\frac{1}{p}\right)^r p\,(x-1)\,(1-p)^{x-2}\,(-1)$$

$$= \frac{r}{p^2}\sum_{x=1}^{\infty}\left(x-\frac{1}{p}\right)^{r-1} p\,(1-p)^{x-1} + \sum_{x=1}^{\infty}\left(x-\frac{1}{p}\right)^r p\,(1-p)^{x-1}\left[\frac{1}{p} - \frac{x-1}{1-p}\right]$$

$$= \frac{r}{p^2}\mu_{r-1} + \sum_{x=1}^{\infty}\left(x-\frac{1}{p}\right)^r p\,(1-p)^{x-1}\left[\frac{1-p-px+1}{p(1-p)}\right]$$

$$= \frac{r}{p^2}\mu_{r-1} + \sum_{x=1}^{\infty}\left(x-\frac{1}{p}\right)^r p\,(1-p)^{x-1}\left[\frac{-p\left(x-\frac{1}{p}\right)}{p\,(1-p)}\right]$$

$$= \frac{r}{p^2}\mu_{r-1} - \frac{1}{q}\sum_{x=1}^{\infty}\left(x-\frac{1}{p}\right)^{r+1} p\,(1-p)^{x-1}$$

$$= \frac{r}{p^2}\mu_{r-1} - \frac{1}{q}\mu_{r+1}$$

$$\therefore \quad \frac{1}{q}\mu_{r+1} = \frac{r}{p^2}\mu_{r-1} - \frac{d}{dp}\mu_r$$

$$\therefore \quad \mu_{r+1} = q\left[\frac{r}{p^2}\mu_{r-1} - \frac{d}{dp}\mu_r\right]$$

$$\therefore \quad \mu_2 = q\left[\frac{1}{p^2}\mu_0 - 0\right] = \frac{q}{p^2}$$

$$\mu_3 = q\left[\frac{2}{p^2}\mu_1 - \frac{d}{dp}\mu_2\right] = -q\frac{d}{dp}\left(\frac{1-p}{p^2}\right) = -q\,\frac{p^2(-1) - (1-p)(2p)}{p^4}$$

$$= -q\cdot\frac{-p^2 - 2p + 2p^2}{p^4}$$

$$= -q\,\frac{p^2 - 2p}{p^4}$$

$$= q\,\frac{(2-p)}{p^3} = \frac{q(1+q)}{p^3}$$

$$\mu_4 = q\left[\frac{3}{p^2}\mu_2 - \frac{d}{dp}\mu_3\right]$$

$$= q\left[\frac{3}{p^2}\cdot\frac{q}{p^2} - \frac{d}{dp}\left\{\frac{(1-p)(2-p)}{p^3}\right\}\right]$$

$$= q\left[\frac{3q}{p^4} - \frac{d}{dp}\left\{\frac{2-3p+p^2}{p^3}\right\}\right]$$

$$= q\left[\frac{3q}{p^4} - \left\{\frac{p^3(-3+2p) - (2-3p+p^2)\,3p^2}{p^6}\right\}\right]$$

$$= q\left[\frac{3q}{p^4} - \left\{\frac{-3p+2p^2-6+9p-3p^2}{p^4}\right\}\right]$$

$$= q\left[\frac{3q}{p^4} - \left(\frac{-p^2+6p-6}{p^4}\right)\right]$$

$$= q\left[\frac{3q+p^2-6(1-q)+6}{p^4}\right] = \frac{q}{p^4}\left(p^2+3q\right)$$

Factorial Moment :-

Factorial moment of order 'r' about zero is

$$\mu'_{[r]}(0) = E\left[X(X-1)(X-2)\cdots(X-r+1)\right]$$

$$= p\sum_{x=1}^{\infty}\left[x(x-1)(x-2)\cdots(x-\overline{r-1})\right]q^{x-r}$$

$$= p\,q^{r-1}\sum_{x=1}^{\infty}\frac{d^r}{dq^r}\left(q^x\right)$$

$$= p\,q^{r-1}\frac{d^r}{dq^r}\left[\sum_{x=1}^{\infty}q^x\right]$$

$$= p\,q^{r-1}\frac{d^r}{dq^r}\left[\frac{q}{1-q}\right] \qquad [\because q<1]$$

$$= p\,q^{r-1}\frac{d^{r-1}}{dq^{r-1}}\left[\frac{1}{(1-q)^2}\right]$$

$$= p\,q^{r-1}\frac{d^{r-2}}{dq^{r-2}}\left[\frac{1.2}{(1-q)^3}\right]$$

$$= p\,q^{r-1}\frac{d^{r-3}}{dq^{r-3}}\left[\frac{1.2.3}{(1-q)^4}\right]$$

$$\vdots$$

$$= p\,q^{r-1}\left[\frac{1.2.3\ldots r}{(1-q)^{r+1}}\right] = \frac{r!\,p\,q^{r-1}}{(1-q)^{r+1}} = \frac{r!\,q^{r-1}}{p^r}$$

For $r = 1, 2, 3, 4$ we have

$$\mu'_{[1]}(0) = \frac{1}{p}$$

$$\mu'_{[2]}(0) = \frac{2q}{p^2}$$

$$\mu'_{[3]}(0) = \frac{3!\,q^2}{p^3} = \frac{6q^2}{p^3}$$

$$\mu'_{[4]}(0) = \frac{4!\,q^3}{p^4} = \frac{24q^3}{p^4}$$

$$\therefore \quad \mu_1'(0) = E(x) = \mu_{[1]}'(0) = \frac{1}{p}$$

$$\mu_2'(0) = E(x^2) = E[x(x-1)+x] = E[x(x-1)] + E(x)$$

$$= \frac{2q}{p^2} + \frac{1}{p} = \frac{2q+p}{p^2} = \frac{1+q}{p^2}$$

$$\mu_3'(0) = E(x^3) = E[x(x-1)(x-2) + 3x(x-1) + x]$$

$$= \mu_{[3]}'(0) + 3\mu_{[2]}'(0) + \mu_{[1]}'(0)$$

$$= \frac{6q^2}{p^3} + 3 \cdot \frac{2q}{p^2} + \frac{1}{p} = \frac{6q^2 + 6pq + p^2}{p^3}$$

$$\mu_4'(0) = E(x^4) = E[x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x]$$

$$= \mu_{[4]}'(0) + 6\mu_{[3]}'(0) + 7\mu_{[2]}'(0) + \mu_{[1]}'(0)$$

$$= \frac{24q^3}{p^4} + 6 \cdot \frac{6q^2}{p^3} + 7\frac{2q}{p^2} + \frac{1}{p}$$

$$= \frac{24q^3 + 36pq^2 + 14p^2q + p^3}{p^4}$$

Then,

$$\mu_2 = \mu_2' - \mu_1'^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= \frac{6q^2+6pq+p^2}{p^3} - 3\left(\frac{1+q}{p^2}\right)\frac{1}{p} + \frac{2}{p^3}$$

$$= \frac{6q^2 + 6pq + p^2 - 3 - 3q + 2}{p^3}$$

$$= \frac{6q^2 + 6(1-q)q + (1-q)^2 - 3 - 3q + 2}{p^3}$$

$$= \frac{6q^2 + 6q - 6q^2 + 1 - 2q + q^2 - 3 - 3q + 2}{p^3} = \frac{q+q^2}{p^3} = \frac{q(1+q)}{p^3}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= \frac{24q^3 + 36pq^2 + 14p^2q + p^3}{p^4} - \frac{4(6q^2+6pq+p^2)}{p^3}\cdot\frac{1}{p} + 6\,\frac{1+q}{p^2}\cdot\frac{1}{p^2} - \frac{3}{p^4}$$

$$= \frac{1}{p^4}\Big[24q^3 + 36(1-q)q^2 + 14(1-2q+q^2)q + (1-3q+3q^2-q^3) - 24q^2 - 24(1-q)q$$
$$- 4(1-2q+q^2) + 6 + 6q - 3\Big]$$

$$= \frac{1}{p^4}\Big[24q^3 + 36q^2 - 36q^3 + 14q - 28q^2 + 14q^3 + 1 - 3q + 3q^2 - q^3 - 24q^2 - 24q$$
$$+ 24q^2 - 4 + 8q - 4q^2 + 6 + 6q - 3\Big]$$

$$= \frac{1}{p^4}\left[q^3 + 7q^2 + q\right]$$

$$= \frac{q}{p^4}\left[q^2 + 7q + 1\right]$$

$$= \frac{q}{p^4}\left(1 - 2p + p^2 + 7 - 7p + 1\right)$$

$$= \frac{q}{p^4}\left(p^2 - 9p + 9\right) = \frac{q(p^2 + 9q)}{p^4}$$

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The coefficient of Skewness & Kurtosis —

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\dfrac{q^2(1+q)^2}{p^6}}{\dfrac{q^3}{p^6}} = \frac{(1+q)^2}{q}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\dfrac{q(p^2+9q)}{p^4}}{\dfrac{q^2}{p^4}} = \frac{p^2+9q}{q} = 9 + \frac{p^2}{q}$$

Skewness

$$\gamma_1 = \sqrt{\beta_1} = \frac{1+q}{\sqrt{q}} > 0$$

Kurtosis

$$\gamma_2 = \beta_2 - 3 = 9 + \frac{p^2}{q} - 3 = 6 + \frac{p^2}{q} > 0$$

$\therefore$ The Distribution is positively skewed and Leptokurtic.

Geometric Distribution (2)

Let X be a random variable denoting the no. of failures preceeding the first success. The probability distribution of X is given by

$$P(X=x) = P\{x \text{ failures will proceed the 1st success}\}$$
$$= P\{\underbrace{F \cap F \cap \cdots \cap F}_{x} \cap S\}$$
$$= q^x p$$

Hence a r.v X will be said to follow geometric distribution of 2nd type with parameters p iff its p.m.f is given by

$$f(x) = q^x p \quad ; \quad x = 0, 1, 2, \ldots$$
$$= 0 \quad ; \quad \text{o.w.}$$

Expectations :-

$$E(X) = \sum_{x=0}^{\infty} x f(x)$$
$$= \sum_{x=0}^{\infty} x q^x p$$
$$= pq \sum_{x=0}^{\infty} x q^{x-1} = pq \sum_{x=0}^{\infty} \frac{d}{dq}(q^x)$$
$$= pq \frac{d}{dq}\left[\sum_{x=0}^{\infty} q^x\right]$$
$$= pq \frac{d}{dq}\left[\frac{1}{1-q}\right] = \frac{pq}{(1-q)^2} = \frac{q}{p}$$

Variance :-

$$V(X) = E(X^2) - E^2(X)$$

Now, $E(X^2) = \sum_{x=0}^{\infty} x^2 f(x)$

$$= \sum_{x=0}^{\infty} x(x-1) f(x) + \sum_{x=0}^{\infty} x f(x)$$
$$= \sum_{x=0}^{\infty} x(x-1) q^x p + E(X)$$
$$= pq^2 \sum_{x=0}^{\infty} x(x-1) q^{x-2} + \frac{q}{p}$$
$$= pq^2 \sum_{x=0}^{\infty} \frac{d^2}{dq^2}(q^x) + \frac{q}{p}$$
$$= pq^2 \frac{d^2}{dq^2}\left[\sum_{x=0}^{\infty} q^x\right] + \frac{q}{p}$$
$$= pq^2 \frac{d^2}{dq^2}\left[\frac{1}{1-q}\right] + \frac{q}{p}$$

$$= pq^2 \frac{d}{dq}\left[\frac{1}{(1-q)^2}\right] + \frac{q}{p}$$

$$= pq^2 \cdot \frac{2}{(1-q)^3} + \frac{q}{p} = \frac{2q^2}{p^2} + \frac{q}{p} = \frac{2q^2+pq}{p^2}.$$

$$\therefore\ V(x) = \frac{2q^2+pq}{p^2} - \frac{q^2}{p^2} = \frac{2q^2+pq-q^2}{p^2} = \frac{q^2+pq}{p^2} = \frac{q}{p^2}.$$

Moment Generation Function (MGF):-

$$M_x(t) = E(e^{xt})$$

$$= \sum_{x=0}^{\infty} e^{tx} f(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} q^x p = p\sum_{x=0}^{\infty}(qe^t)^x = \frac{p}{1-qe^t} \qquad [\because qe^t < 1]$$

$$\frac{d}{dt}M_x(t)\Big|_{t=0} = \frac{d}{dt}\left(\frac{p}{1-qe^t}\right)\Big|_{t=0} = \frac{(1-qe^t).0 + p(1-qe^t)(qe^t)}{(1-qe^t)^2}\Big|_{t=0}$$

$$= \left[\frac{pqe^t}{(1-qe^t)^2}\right]_{t=0} = \frac{pq}{(1-q)^2} = \frac{q}{p}$$

$$\frac{d^2}{dt^2}M_x(t)\Big|_{t=0} = \frac{d}{dt}\left[\frac{pqe^t}{(1-qe^t)^2}\right]_{t=0}$$

$$= \left[\frac{(1-qe^t)^2 pqe^t - pqe^t(-2qe^t)(1-qe^t)}{(1-qe^t)^4}\right]_{t=0}$$

$$= \frac{(1-q)^2 pq + 2pq^2(1-q)}{(1-q)^4}$$

$$= \frac{pq}{(1-q)^2} + \frac{2pq^2}{(1-q)^3} = \frac{q}{p} + \frac{2q^2}{p^2}$$

$$\therefore\ V(x) = \frac{q}{p} + \frac{2q^2}{p^2} - \frac{q^2}{p^2} = \frac{q}{p} + \frac{q^2}{p^2} = \frac{q(p+q)}{p^2} = \frac{q}{p^2}$$

Probability Generating Function (PGF)

$$P_x(t) = E(t^x)$$

$$= \sum_{x=0}^{\infty} t^x q^x p$$

$$= p\sum_{x=0}^{\infty}(qt)^x = \frac{p}{1-qt}, \qquad |t|<1$$

Recursion Relation :-

$$\mu_r = E\left(x - \frac{q}{p}\right)^r \quad , \text{ here } \mu = \frac{q}{p}$$

$$= \sum_{x=0}^{\infty} \left(x - \frac{q}{p}\right)^r p\, q^x \qquad = \sum_{x=0}^{\infty} \left(x - \frac{1-p}{p}\right)^r p\,(1-p)^x$$

$$\therefore \frac{d}{dp}\mu_r = r \sum_{x=0}^{\infty} \left(x - \frac{1-p}{p}\right)^{r-1} \left[- \frac{p(-1) - (1-p)}{p^2}\right] p\,(1-p)^x$$

$$+ \sum_{x=0}^{\infty} \left(x - \frac{1-p}{p}\right)^r (1-p)^x + \sum_{x=0}^{\infty} \left(x - \frac{1-p}{p}\right)^r p\; x(1-p)^{x-1}(-1)$$

$$= \frac{r}{p^2} \sum_{x=0}^{\infty} \left(x - \frac{1-p}{p}\right)^{r-1} p\,(1-p)^x$$

$$+ \sum_{x=0}^{\infty} \left(x - \frac{1-p}{p}\right)^r p\,(1-p)^x \left[\frac{1}{p} - \frac{x}{1-p}\right]$$

$$= \frac{r}{p^2}\mu_{r-1} - \frac{1}{1-p} \sum_{x=0}^{\infty} \left(x - \frac{1-p}{p}\right)^r p\,(1-p)^x \left(x - \frac{1-p}{p}\right)$$

$$= \frac{r}{p^2}\mu_{r-1} - \frac{1}{1-p} \sum_{x=0}^{\infty} \left(x - \frac{1-p}{p}\right)^{r+1} p\,(1-p)^x$$

$$= \frac{r}{p^2}\mu_{r-1} - \frac{1}{q}\mu_{r+1}$$

$$\Rightarrow \frac{1}{q}\mu_{r+1} = \frac{r}{p^2}\mu_{r-1} - \frac{d}{dp}\mu_r$$

$$\therefore \mu_{r+1} = q\left[\frac{r}{p^2}\mu_{r-1} - \frac{d}{dp}\mu_r\right]$$

$$\therefore \mu_2 = \frac{q}{p^2}$$

$$\mu_3 = q\left[-\frac{d}{dp}\mu_2\right] = q\left[-\frac{d}{dp}\left(\frac{1-p}{p^2}\right)\right] = q\left[-\left\{\frac{p^2(-1) - (1-p)2p}{p^4}\right\}\right]$$

$$= q\,\frac{2p - p^2}{p^4} = \frac{q(2-p)}{p^3} = \frac{q(1+q)}{p^3}$$

$$\mu_4 = q\left[\frac{3}{p^2}\mu_2 - \frac{d}{dp}\mu_3\right]$$

$$= q\left[\frac{3}{p^2}\cdot\frac{q}{p^2} - \frac{d}{dp}\,\frac{(1-p)(2-p)}{p^3}\right]$$

$$= q\left[\frac{3q}{p^4} - \frac{-p^2 + 6p - 6}{p^4}\right]$$

$$= q\left[\frac{3q + p^2 - 6(1-q) - 6}{p^4}\right] = \frac{q}{p^4}\left(p^2 + 9q\right)$$

$$\therefore \beta_1 = \frac{(1+q)^2}{q} \quad , \quad \beta_2 = 9 + \frac{p^2}{q}$$

$$\gamma_1 = \frac{1+q}{\sqrt{q}} \quad , \quad \gamma_2 = 6 + \frac{p^2}{q}$$

Negative Binomial Distribution

Let X be a r.v denoting the no. of failures preceeding the 1st k successes. The probability distribution is defined as

$$f(x) = P[X=x] = \text{Prob.}\{x \text{ failures preceeding the } k \text{ successes}\}$$
$$= \binom{x+k-1}{k-1} p^{k-1} q^{x} p$$
$$= \binom{x+k-1}{k-1} p^{k} q^{x}$$

Hence a distribution is said to be negative binomial if its p.m.f is given by

$$f(x) = \binom{x+k-1}{k-1} p^{k} q^{x} \quad ; x = 0, 1, 2, \ldots$$
$$= 0 \quad ; \text{o.w.}$$

Expectation :-

$$E(X) = \sum_{x=0}^{\infty} x f(x)$$
$$= \sum_{x=0}^{\infty} x \binom{x+k-1}{k-1} p^{k} q^{x}$$
$$= p^{k} \sum_{x=0}^{\infty} x \, \frac{(x+k-1)(x+k-2)\cdots k}{x!} q^{x}$$
$$= k p^{k} q \sum_{x=1}^{\infty} \frac{(x+k-1)(x+k-2)\cdots(k+1)}{(x-1)!} q^{x-1}$$
$$= k p^{k} q (1-q)^{-(k+1)}$$
$$= \frac{kq}{p}$$

$$\left[\sum_{x=0}^{\infty} f(x) = p^{k} \sum_{x=0}^{\infty} \frac{(k+x-1)\cdots(k)}{x!} q^{x} = p^{k}(1-q)^{-k} = 1 \right]$$

Variance :-

$$V(X) = E(X^2) - E^2(X)$$
$$E(X^2) = E[X(X-1) + X] = E[X(X-1)] + E(X)$$

Now
$$E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1) \binom{x+k-1}{k-1} p^{k} q^{x}$$
$$= p^{k} \sum_{x=0}^{\infty} x(x-1) \frac{(x+k-1)(x+k-2)\cdots k}{x!} q^{x}$$
$$= p^{k} k(k+1) q^{2} \sum_{x=2}^{\infty} \frac{(x+k-1)(x+k-2)\cdots(k+2)}{(x-2)!} q^{x-2}$$
$$= k(k+1) p^{k} q^{2} (1-q)^{-(k+2)} = \frac{k(k+1)q^2}{p^2}$$

$$\therefore V(X) = \frac{k(k+1)q^2}{p^2} + \frac{kq}{p} - \frac{k^2q^2}{p^2}$$
$$= \frac{k^2q^2}{p^2} + \frac{kq^2}{p^2} + \frac{kq}{p} - \frac{k^2q^2}{p^2} = \frac{kq^2}{p^2} + \frac{kq}{p} = \frac{kq(q+p)}{p^2} = \frac{kq}{p^2}.$$

Moment Generating Function (MGF) :-

$$M_X(t) = E\left[e^{xt}\right]$$
$$= \sum_{x=0}^{\infty} e^{xt} \binom{x+k-1}{k-1} p^k q^x$$
$$= p^k \sum_{x=0}^{\infty} \frac{(x+k-1)(x+k-2)\cdots k}{x!} (e^t q)^x$$
$$= p^k (1-qe^t)^{-k} \qquad = \left[\frac{p}{1-qe^t}\right]^k$$

Probability Generating Function (PGF) :-

$$P_X(t) = E(t^X)$$
$$= \sum_{x=0}^{\infty} t^x \binom{x+k-1}{k-1} p^k q^x$$
$$= p^k \sum_{x=0}^{\infty} \frac{(x+k-1)(x+k-2)\cdots k}{x!} (qt)^x$$
$$= p^k (1-tq)^{-k} \qquad = \left[\frac{p}{1-tq}\right]^k$$

Recursion Relation :-

$$\mu_r = E(X-\mu)^r$$
$$= E\left(X - \frac{kq}{p}\right)^r$$
$$= \sum_{x=0}^{\infty} \left(x - \frac{kq}{p}\right)^r \binom{x+k-1}{k-1} p^k q^x$$
$$= \sum_{x=0}^{\infty} \binom{x+k-1}{k-1} \left[x - \frac{k(1-p)}{p}\right]^r p^k (1-p)^x$$

$$\frac{d}{dp}\mu_r = \frac{d}{dp}\left[\sum_{x=0}^{\infty} \binom{x+k-1}{k-1} \left\{x - \frac{k(1-p)}{p}\right\}^r p^k (1-p)^x\right]$$
$$= \sum_{x=0}^{\infty} \binom{x+k-1}{k-1} \frac{d}{dp}\left[\left\{x - \frac{k(1-p)}{p}\right\}^r p^k (1-p)^x\right]$$
$$= \sum_{x=0}^{\infty} \binom{x+k-1}{k-1} \left[r\left\{x - \frac{k(1-p)}{p}\right\}^{r-1} \frac{k}{p^2} p^k (1-p)^x + kp^{k-1}\left\{x - \frac{k(1-p)}{p}\right\}^r (1-p)^x - x(1-p)^{x-1}\left\{x - \frac{k(1-p)}{p}\right\}^r p^k\right]$$
$$= \frac{rk}{p^2}\mu_{r-1} + \sum_{x=0}^{\infty} \binom{x+k-1}{k-1} p^k (1-p)^k \left\{x - \frac{k(1-p)}{p}\right\}^r \left(\frac{k}{p} - \frac{x}{1-p}\right)$$
$$= \frac{rk}{p^2}\mu_{r-1} - \sum_{x=0}^{\infty} \left\{x - \frac{k(1-p)}{p}\right\}^r \binom{x+k-1}{k-1} p^k (1-p)^k \frac{1}{1-p}\left\{x - \frac{k(1-p)}{p}\right\}$$
$$= \frac{rk}{p^2}\mu_{r-1} - \frac{1}{1-p}\sum_{x=0}^{\infty} \left\{x - \frac{k(1-p)}{p}\right\}^{r+1} \binom{x+k-1}{k-1} p^k (1-p)^k$$
$$= \frac{rk}{p^2}\mu_{r-1} - \frac{1}{q}\mu_{r+1}$$

$$\therefore \frac{1}{q}\mu_{r+1} = \frac{rk}{p^2}\mu_{r-1} - \frac{d}{dp}\mu_r$$

$$\Rightarrow \mu_{r+1} = q\left[\frac{rk}{p^2}\mu_{r-1} - \frac{d}{dp}\mu_r\right]$$

$$\therefore \mu_1 = 0$$

$$\mu_2 = \frac{kq}{p^2}$$

For $r = 3, 4$ we have

$$\mu_3 = q\left[\frac{2k}{p^2}\mu_1 - \frac{d}{dp}\mu_2\right]$$

$$= q\left[-\frac{d}{dp}\left\{\frac{k(1-p)}{p^2}\right\}\right]$$

$$= q\left[-\frac{p^2(-k) - k(1-p)2p}{p^4}\right]$$

$$= q\left[\frac{kp^2 + 2pk - 2kp^2}{p^4}\right]$$

$$= \frac{qk}{p^4}\left[\frac{2p - p^2}{p^4}\right]$$

$$= \frac{qk}{p^3}(2-p) = \frac{qk(1+q)}{p^3}$$

$$\mu_4 = q\left[\frac{3k}{p^2}\mu_2 - \frac{d}{dp}\mu_3\right]$$

$$= q\left[\frac{3k}{p^2}\frac{kq}{p^2} - \frac{d}{dp}\left\{\frac{qk(2-p)}{p^3}\right\}\right]$$

$$= q\left[\frac{3k^2q}{p^4} - \frac{d}{dp}\left\{\frac{k(1-p)(2-p)}{p^3}\right\}\right]$$

$$= q\left[\frac{3k^2q}{p^4} - k\frac{d}{dp}\left(\frac{2-3p+p^2}{p^3}\right)\right]$$

$$= q\left[\frac{3k^2q}{p^4} - k\frac{d}{dp}\left(\frac{2}{p^3} - \frac{3}{p^2} + \frac{1}{p}\right)\right]$$

$$= q\left[\frac{3k^2q}{p^4} - k\left(-\frac{6}{p^4} + \frac{6}{p^3} - \frac{1}{p^2}\right)\right]$$

$$= q\left[\frac{3k^2q}{p^4} - k\frac{(-6+6p-p^2)}{p^4}\right]$$

$$= \frac{qk}{p^4}\left[3kq - \{-6q - (1-2q+q^2)\}\right]$$

$$= \frac{qk}{p^4}\left[3kq - (-4q-1-q^2)\right]$$

$$= \frac{qk}{p^4}\left[3kq + 1 + 4q + q^2\right]$$

$\therefore$ Skewness :

$$\gamma_1 = \sqrt{\beta_1} = \sqrt{\frac{\mu_3^2}{\mu_2^3}}$$

$$= \sqrt{\frac{\frac{q^2k^2(1+q)^2}{p^6}}{\frac{k^3q^3}{p^6}}} = \sqrt{\frac{(1+q)^2}{kq}} = \frac{1+q}{\sqrt{kq}} > 0$$

Kurtosis :

$$\gamma_2 = \beta_2 - 3$$

$$= \frac{\mu_4}{\mu_2^2} - 3$$

$$= \frac{\frac{qk}{p^4}(3kq + 1 + 4q + q^2)}{\frac{q^2k^2}{p^4}} - 3$$

$$= \frac{3kq + 1 + 4q + q^2}{qk} - 3$$

$$= \frac{1 + 4q + q^2}{qk} > 0$$

$\therefore$ The distribution is positively skewed and Leptokurtic.

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