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NAAC ACCREDITED 'A' GRADE

Topic: Series of Real numbers

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4. Series of numbers

Given a sequence $\{x_n\}$ of real numbers, we associate with it the infinite sum or series denoted by $\sum_{n=1}^{\infty} x_n$. Our goal here is to assign a precise meaning to the symbol $\sum_{n=1}^{\infty} x_n$. Intuitively, we understand that $\sum_{n=1}^{\infty} x_n$ means the value of the finite sum $\sum_{n=1}^{k} x_n$ for larger and larger values of k and this intuition leads to the following idea of convergence of series.

About Convergent and Divergent series

Given a sequence $\{x_n\}$, consider the finite sums of the type $S_k = \sum_{n=1}^{k} x_n$, for $k \in \mathbb{N}$. Then we say that the series $\sum_{n=1}^{\infty} x_n$ converges to a number l and write $\sum_{n=1}^{\infty} x_n = l$ if the sequence $\{S_k\}$ converges to l as $k \to \infty$. If this does not happen then we say that the series is divergent and the symbol $\sum_{n=1}^{\infty} x_n$ has no meaning. However, if $\lim_{k\to\infty} S_k$ is positive or negative infinity, then we say that the series $\sum_{n=1}^{\infty} x_n$ diverges to infinity and write $\sum_{n=1}^{\infty} x_n = \infty$ or $\sum_{n=1}^{\infty} x_n = -\infty$ respectively.

The sequence $\{S_k\}$ is often called the sequence of partial sums of the series $\sum_{n=1}^{\infty} x_n$. We have earlier made a point that the convergence or divergence of any sequence is not affected by the first million or the first billion terms of the sequence; so this applies to the sequence $\{S_k\}$ and consequently the convergence or divergence of any series is not affected by the first million or first billion terms of the series.

Notice that if the series $\sum_{n=1}^{\infty} x_n$ converges to a number l then

$\lim_{k\to\infty} x_k = \lim_{k\to\infty} (S_k - S_{k-1}) = l - l = 0.$

Thus for the series $\sum_{n=1}^{\infty} x_n$, if $\lim_{k\to\infty} x_k \neq 0$, then we can immediately conclude that the series is not convergent. On the other hand if $\lim_{k\to\infty} x_k = 0$, then the series may not be necessarily convergent. For this, we may consider the example of the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$. it can be worked out that if $S_k = \sum_{n=1}^{k}\frac{1}{n}$, then $S_{2^k} \geq \frac{k+2}{2}$ and so the sequence $\{S_k\}$ turns out to be unbounded. This results into $\{S_k\}$ being divergent and hence the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is not convergent. In fact, we have $\sum_{n=1}^{\infty}\frac{1}{n} = \infty$. this can also be interpreted as; given any positive number M we can find a positive integer k such that $\sum_{n=1}^{k}\frac{1}{n} > M$.

5. Basic properties of the series

1) If the series $\sum_{n=1}^{\infty} x_n$ and the series $\sum_{n=1}^{\infty} y_n$ are convergent then the series $\sum_{n=1}^{\infty}(x_n \pm y_n)$ is convergent and

$$\sum_{n=1}^{\infty}(x_n \pm y_n) = \sum_{n=1}^{\infty} x_n \pm \sum_{n=1}^{\infty} y_n.$$

2) If the series $\sum_{n=1}^{\infty} x_n$ converges and $\alpha \in \mathbb{R}$, then the series $\sum_{n=1}^{\infty}(\alpha x_n)$ converges and

$$\sum_{n=1}^{\infty} \alpha x_n = \alpha \sum_{n=1}^{\infty} x_n.$$

6. Geometric series

Let r be a real number. Then the series $\sum_{n=1}^{\infty} r^n = r + r^2 + r^3 + \cdots$ is called the geometric series. The test for the convergence of geometric series is very simple. If $|r| < 1$ then it is convergent and further $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$ whereas if $|r| \geq 1$ then it is divergent.

Such type of series occurs when we think about the decimal expansions of some rational numbers. For instance, if we consider the number 1/3 its decimal expansion is 0.3333..... which may be

$$3\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots\right)=3\left(\frac{1/10}{1-1/10}\right)=\frac{3}{9}=\frac{1}{3}.$$

7. Telescoping series

A telescoping series is a series whose partial sums eventually have fixed number of terms after cancellation. One popular example of such a series is

$$\sum_{n=1}^{\infty}\frac{1}{n\,(n+1)} \quad \text{or more generally} \sum_{n=1}^{\infty}\frac{1}{(n+l)\,(n+p)} \text{ where } l<p.$$

Here we may write

$$\sum_{n=1}^{\infty}\frac{1}{(n+l)\,(n+p)}=\frac{1}{p-l}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n+l}-\frac{1}{n+p}\right)\right)$$

It is very easy to determine the convergence of the series which are geometric or telescoping but this is not always the case for Arbitrary series. So we now introduce some of the tests which help us in determining the convergence or divergence of a given series.

8. Comparison test

1) If $|x_n|\le y_n$ for $n\ge n_0$, and if $\sum y_n$ converges, then $\sum x_n$ converges.
2) If $x_n\ge y_n\ge 0$ for $n\ge n_0$, and if $\sum y_n$ diverges then $\sum x_n$ diverges.

The comparison test is very useful especially if we are familiar with certain convergent or divergent series of non-negative numbers.

9. Cauchy's Condensation test

Let $\{x_n\}$ be a decreasing sequence of non-negative terms. Then the series $\sum x_n$ converges if and only if the series $\sum_{n=1}^{\infty}2^n x_{2^n}$ converges. In view of the geometric series test, the Cauchy's condensation test now immediately gives some nice results.

Results

1) The series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

2) The series $\sum \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

Using these results we now work out an example based on the comparison test.

Example: Determine the convergence or divergence of the series $\sum \sin n^2/n^2$ and $\sum \frac{\log n}{n}$.

Solution: Let $x_n = \sin n^2/n^2$ and $y_n = 1/n^2$. since $|x_n| \leq y_n$ for all $n \geq 1$ and because we know that $\sum 1/n^2$ is a convergent series, it follows from the comparison test that the series $\sum \frac{\sin n^2}{n^2}$ converges. For the second series we take $x_n = \log n/n$ and $y_n = 1/n$. Note that $x_n \geq y_n \geq 0$ for large values of n and further the harmonic series $\sum 1/n$ is divergent. So the comparison test implies that the series $\sum \frac{\log n}{n}$ diverges.

10. The Alternating series test

Let $\{x_n\}$ be a decreasing sequence of non-negative terms. Then the series $\sum(-1)^n x_n$ converges if $\lim_{n\to\infty} x_n = 0$.

Recall that $\sum \frac{1}{n^p}$ is divergent for $0 < p \leq 1$ follows from the Cauchy condensation test. But if we alternate the sign of the terms of this series then due to the alternating series test we deduce that $\sum(-1)^n/n^p$ is convergent for any $p > 0$. this result indicates that sometimes the convergence of a series is greatly influenced by the cancellations involved in the series.

The comparison test suggests that a series converges if the rate of convergence of its terms to zero is high enough. But it is not always possible to judge this rate of convergence and in such scenarios, sometimes the ratio and the root tests are useful. So we discuss these tests one by one.

The Ratio test

For the series $\sum x_n$, suppose that $\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = l$. then

1) $\sum x_n$ converges if $l < 1$;
2) $\sum x_n$ diverges if $l > 1$;
3) the test is inconclusive if $l = 1$.

The ratio test can be applied to conclude that the series $\sum x^n/n!$ converges for every real x. Further; it can also be used to establish that the series $\sum n!/n^n$ converges.

The ratio test is mainly used when the general term of the series involve factorials. Some of the series where this test is inconclusive are $\sum 1/n^p$ and $\sum \frac{n^2}{n^2+2n}$.

The Root test

For the series $\sum x_n$, suppose that $\lim_{n\to\infty}|x_n|^{1/n} = l$. Then

1) $\sum x_n$ converges if $l < 1$;
2) $\sum x_n$ diverges if $l > 1$;
3) the test is inconclusive if $l = 1$.

Here we are assessing the value of the n^{th} root and so the root test is mainly used when the general term of the series has powers. The following rules for non-exponentials are also sometimes useful while applying the root test.

1) $\lim_{n\to\infty} c^{1/n} = 1$ for every positive constant c.
2) $\lim_{n\to\infty} (\log n)^{1/n} = 1$.
3) $\lim_{n\to\infty} (n^p)^{1/n} = 1$ for every positive exponent p.
4) $\lim_{n\to\infty} (n!)^{1/n} = \infty$.

Example: Test the convergence or divergence of the series $\sum(-1)^n n^2/(1+n)^n$ and $\sum\frac{2^{n^2}}{n^n}$.

Solution: Let $x_n = \frac{(-1)^n n^2}{(1+n)^n}$. Then

$$\lim|x_n|^{\frac{1}{n}} = \lim\frac{\left(n^{\frac{1}{n}}\right)^2}{1+n} = \lim\frac{1}{1+n} = 0 < 1.$$

Hence by the root test the series $\sum(-1)^n n^2/(1+n)^n$ is convergent. For the second series we observe that

$$\lim|x_n|^{\frac{1}{n}} = \lim\frac{2^n}{n} = \infty$$

And hence the series $\sum\frac{2^{n^2}}{n^n}$ is divergent.

11. Summary

We give a formal definition of a sequence and its limit and understand the concepts related to the convergence and divergence of sequences. Later, we use the concept of a limit of a sequence to define the infinite sums or series. The problem of determining whether a given series is convergent or divergent is not always easy and for this various tests are introduced and studied in this module.