VIVEKANANDA COLLEGE
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NAAC ACCREDITED 'A' GRADE

Topic: Discrete Uniform_Binomial_Poisson Distribution

Course Title: Discrete Probability Distribution

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where getting head ... as a success with probability ½ in each trial.

SOME DISCRETE DISTRIBUTIONS

12.3 The uniform distribution for discrete variable

This distribution occurs when the different values of the random variable are equally probable. Suppose, for instance, that an unbiased die is rolled and the random variable X denotes the number of points on the uppermost face of the die. Then X has a uniform distribution, because it takes the values 1, 2, ..., 6, each with probability 1/6. In general, this distribution is defined by the probability-mass function.

$$f(x) = \frac{1}{n}, \qquad x = a,\ a + h,\ a + 2h,\ \ldots,\ a + (n - 1)h,$$
$$= 0, \qquad \text{elsewhere,}$$

where a and h are fixed real numbers and n is a fixed positive integer.

Obviously, $\qquad f(x) \geq 0$, for all x,

$$\text{and } \sum_x f(x) = \sum_x \frac{1}{n} = n \times \frac{1}{n} = 1.$$

12.3.1 Important properties of the distribution

Before finding the different measures, we note the following sums that will be used.

$$\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}, \quad \sum_{i=0}^{n-1} i^2 = \frac{1}{6}n(n-1)(2n-1),$$

$$\sum_{i=0}^{n-1} i^3 = \left\{\frac{n(n-1)}{2}\right\}^2 \text{ and } \sum_{i=0}^{n-1} i^4 = \frac{1}{30}n(n-1)(2n-1)(3n^2-3n-1).$$

For this distribution,

$$\text{mean} = \mu = E(X) = \sum_x x f(x) = \frac{1}{n}\sum_{i=0}^{n-1}(a+ih) = \frac{1}{n}\left[na + h\sum_{i=0}^{n-1} i\right]$$

$$= \frac{1}{n}[na + h.\ \frac{1}{2}\ n(n-1)] = a + \frac{1}{2}h(n-1).$$

$$\text{Var}(X) = \sigma^2 = E(X-\mu)^2 = \sum_x (x-\mu)^2 f(x)$$

$$= \frac{1}{n}\sum_{i=0}^{n-1}[(a+ih) - \{a + \frac{1}{2}h(n-1)\}]^2 = \frac{h^2}{n}\sum_{i=0}^{n-1}\left\{i - \frac{1}{2}(n-1)\right\}^2$$

$$= \frac{h^2}{n}\left\{\sum_{i=0}^{n-1} i^2 - (n-1)\sum_{i=0}^{n-1} i + \frac{n(n-1)^2}{4}\right\}$$

$$= \frac{h^2}{n}\left\{\frac{1}{6}n(n-1)(2n-1) - \frac{1}{2}n(n-1)^2 + \frac{1}{4}n(n-1)^2\right\}$$

$$= h^2\left\{\frac{1}{6}(n-1)(2n-1) - \frac{1}{4}(n-1)^2\right\}$$

$$= \frac{h^2}{12}.\ (n-1)\{2(2n-1) - 3(n-1)\} = \frac{h^2}{12}(n^2-1).$$

For measures of skewness and kurtosis, we require third and fourth central moments. Proceeding as before,

$$\mu_3 = E(X-\mu)^3 = \frac{h^3}{n}\sum_{i=0}^{n-1}\left\{i - \frac{1}{2}(n-1)\right\}^3$$

$$= \frac{h^3}{n}\left\{\sum_{i=0}^{n-1} i^3 - \frac{3}{2}(n-1)\sum_{i=0}^{n-1} i^2 + \frac{3}{4}(n-1)^2\sum_{i=0}^{n-1} i - \frac{1}{8}n(n-1)^3\right\}$$

Putting the values of different sums and simplifying, we get

$$\mu_3 = 0.$$

Similarly,

$$\mu_4 = E(X - \mu)^4 = \frac{h^4}{n} \sum_{i=0}^{n-1} \left\{ i - \frac{1}{2}(n-1) \right\}^4 = \frac{h^4}{240}(n^2 - 1)(3n^2 - 7).$$

$$\therefore \ \beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0,$$

and $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{144}{240} \times \dfrac{(3n^2 - 7)}{(n^2 - 1)} = 1.8 \times \dfrac{n^2 - \frac{7}{3}}{n^2 - 1}$ which is less than 1.8 for finite values of n and tends to 1.8 as $n \to \infty$.

So, measure of skewness $= \gamma_1 = \sqrt{\beta_1} = 0$,

and measure of kurtosis $= \gamma_2 = \beta_2 - 3 < 0$.

Thus, the distribution is symmetrical (which is, of course, obvious from the p.m.f.) and platykurtic.

Discrete Distribution

Binomial Distribution :

Let us consider n independent bernoullian trials. Probability of success in each trial is p. Let X be a random variable denoting the number of success obtained in those n trials. Find p.m.f of X.

Sol^n: Let $f(x)$ be the p.m.f of x

$$f(x) = P[X=x] = \text{Prob.}\{x \text{ success will be obtained in } n \text{ trials}\}$$
$$= P(A) \quad [\text{say}]$$

The event A can occur in the following mutually exclusive ways.

We may have a situation

$$A = \{\underbrace{S \cap S \cap \cdots \cap S}_{x} \cap \underbrace{F \cap F \cap \cdots \cap F}_{n-x}\} \cup$$
$$\{\underbrace{S \cap S \cap \cdots \cap S}_{x-1} \cap \underbrace{F}_{1} \cap \underbrace{S}_{1} \cap \underbrace{F \cap F \cap \cdots \cap F}_{n-x+1}\} \cup \cdots$$
$$\cup \{\underbrace{F \cap F \cap \cdots \cap F}_{n-x} \cap \underbrace{S \cap S \cap \cdots \cap S}_{x}\}$$

From theorem of total probability we have,

$$P(A) = P(S \cap S \cap \cdots \cap S \cap F \cap \cdots \cap F) + \cdots + P(F \cap F \cap \cdots \cap F \cap S \cap \cdots \cap S)$$

From theorem of compound probability

$$P(A) = P(S)\cdot P(S) \cdots P(S)\cdot P(F) \cdots P(F) + \cdots + P(F)\cdot P(F) \cdots P(F)\cdot P(S) \cdots P(S)$$
$$= p^x(1-p)^{n-x} + \cdots + p^x(1-p)^{n-x}$$
$$= \frac{\lfloor n}{\lfloor x \lfloor n-x} p^x q^{n-x} \qquad \left[\text{where } q = 1-p \text{ and } \frac{\lfloor n}{\lfloor x \lfloor n-x} = \text{Arrangement of } P(S) \text{ \& } P(F)\right.$$
$$= \binom{n}{x} p^x q^{n-x}$$

$\therefore$ A random variable X is said to follow a binomial dist^n iff its p.m.f can be written as

$$f(x) = \binom{n}{x} p^x q^{n-x} \quad ; \quad x = 0, 1, \cdots, n$$
$$= 0 \quad ; \quad \text{o.w}$$

Expectation :-

$$E(x) = \sum_{x=0}^{n} x \, P[X=x]$$

$$= \sum_{x=0}^{n} x f(x) \qquad \left[\sum f(x) = \binom{n}{0} q^n + \binom{n}{1} p q^{n-1} + \dots + \binom{n}{n} p^n = (p+q)^n = 1 \right]$$

$$= \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{\lfloor\underline{n}}{\lfloor\underline{x} \; \lfloor\underline{n-x}} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{\lfloor\underline{n}}{\lfloor\underline{x-1} \; \lfloor\underline{n-1-x-1}} p^x q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{\lfloor\underline{n-1}}{\lfloor\underline{x-1} \; \lfloor\underline{n-1-\overline{x-1}}} p^{x-1} q^{n-1-\overline{x-1}}$$

$$= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{n-1-\overline{x-1}}$$

$$= np$$

Variance

$$V(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = E[x(x-1) + x]$$

$$= E[x(x-1)] + E(x)$$

$$E[x(x-1)] = \sum_{x=0}^{n} x(x-1) f(x)$$

$$= \sum_{x=0}^{n} x(x-1) \frac{\lfloor\underline{n}}{\lfloor\underline{x} \; \lfloor\underline{n-x}} p^x q^{n-x}$$

$$= \sum_{x=2}^{n} \frac{\lfloor\underline{n}}{\lfloor\underline{x-2} \; \lfloor\underline{n-x}} p^x q^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{\lfloor\underline{n-2}}{\lfloor\underline{x-2} \; \lfloor\underline{n-2-\overline{x-2}}} p^{x-2} q^{n-2-\overline{x-2}}$$

$$= n(n-1) p^2 \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-2-\overline{x-2}}$$

$$= n(n-1) p^2$$

$$\therefore \quad E(x^2) = n(n-1)p^2 + np$$

$$\therefore \quad V(x) = n(n-1)p^2 + np - n^2p^2$$

$$= -np^2 + np$$

$$= np(1-p)$$

$$= npq \qquad [\because q = 1-p]$$

Mode of a Binomial Distribution :-

Let h be the mode of the binomial distⁿ so h must have to satisfy the following two inequalities.

$$f(h) \geq f(h+1) \quad \text{--- (1)}$$
$$f(h) \geq f(h-1) \quad \text{--- (2)}$$

$$\frac{f(h)}{f(h+1)} = \frac{\lfloor n}{\lfloor h \lfloor n-h} p^h q^{n-h} \Big/ \frac{\lfloor n}{\lfloor h+1 \lfloor n-h-1} p^{h+1} q^{n-h-1}$$
$$= \frac{h+1}{n-h} q/p$$

From (1) we have

$$\frac{f(h)}{f(h+1)} \geq 1$$
$$\Rightarrow (h+1)q \geq p(n-h)$$
$$\Rightarrow hq + hp \geq np - q \quad \Rightarrow h \geq np - (1-p)$$
$$\Rightarrow h \geq (n+1)p - 1 \quad \text{--- (3)}$$

$$\therefore \frac{f(h)}{f(h-1)} = \frac{\frac{\lfloor n}{\lfloor h \lfloor n-h} p^h q^{n-h}}{\frac{\lfloor n}{\lfloor h-1 \lfloor n-h+1} p^{h-1} q^{n-h+1}}$$
$$= \frac{n-h+1}{h} p/q$$

From (2) we know that

$$\frac{f(h)}{f(h-1)} \geq 1$$

i.e. $np - hp + p \geq hq$

i.e. $(n+1)p \geq h$ --- (4)

Combining (3) & (4) we have

$$(n+1)p - 1 \leq h \leq (n+1)p$$

Note that h is a point of binomial distⁿ so h must have to be an integer.

There may be two cases —

Case-I

$(n+1)p$ is an integer then obviously $(n+1)p-1$ is also an integer. Since '=' hold both the 2 sides and ∃ no integer in the interval $(n+1)p-1$ upto $(n+1)p$. Obviously h will take 2 values one of them is $(n+1)p$ and other one is $(n+1)p-1$. That is in this case the binomial distⁿ is bi-modal.

Case - II

$(n+1)p$ is not an integer so obviously $(n+1)p-1$ is also not an and there is only one integer in the interval $(n+1)p-1$ and $(n+1)p$ and that integer is $[(n+1)p]$ where $[x]$ is the greatest integer in constant x i.e. $k = [(n+1)p]$. So the Binomial distⁿ in this case is unimodal.

Mean Deviation about Mean

$$MD_{\bar{x}} = \frac{1}{n} \sum_{i=1}^{k} |x_i - \bar{x}| f_i$$

Mean deviation about mean denoted by MD_μ and defined by

$$MD_\mu = E|X-\mu|$$

$$= \sum_{x=0}^{n} |x - np| f(x) \quad \text{——— (1)}$$

Define $\nu = [np]+1$ where $[np]$ is the largest value of x.

From (1)

$$MD_\mu = \sum_{x=0}^{\nu-1} (np - x) f(x) + \sum_{x=\nu}^{n} (x-np) f(x)$$

We know

$$E(x-\mu) = 0$$

$$\Leftrightarrow \quad E(x-np) = 0$$

$$\Rightarrow \quad \sum_{x=0}^{n} (x-np) f(x) = 0$$

$$\Rightarrow \quad \sum_{x=0}^{n} x f(x) - np \sum_{x=0}^{n} f(x) = 0$$

$$\Rightarrow \quad E(X) - np = 0$$

$$\Rightarrow \quad E(X) = np$$

[Expectation of the deviations from the mean is zero]

$$E(X-np) = 0$$

$$\Rightarrow \sum_{x=0}^{\nu-1} (x-np) f(x) + \sum_{x=\nu} (x-np) f(x) = 0$$

$$\Rightarrow \sum_{x=0}^{\nu-1} (x-np) f(x) = - \sum_{x=\nu}^{n} (x-np) f(x)$$

$$\Rightarrow \sum_{x=0}^{\nu-1} (np-x) f(x) = \sum_{x=\nu}^{n} (x-np) f(x)$$

Then

$$MD_\mu = \sum_{x=0}^{\nu-1} (np-x) f(x) + \sum_{x=\nu}^{n} (x-np) f(x)$$

$$= 2 \sum_{x=\nu}^{n} (x-np) f(x)$$

$$= 2 \sum_{x=\nu}^{n} (x-np) \binom{n}{x} p^x q^{n-x}$$

$$= 2\sum_{x=\nu}^{n}(xp+xq-np)\binom{n}{x}p^x q^{n-x}$$

$$= 2\sum_{x=\nu}^{n}(xq-p(n-x))\frac{\lfloor n}{\lfloor x\,\lfloor n-x}p^x q^{n-x}$$

$$= 2\sum_{x=\nu}^{n}\left[\frac{\lfloor n}{\lfloor x-1\,\lfloor n-x}p^x q^{n-x+1}-\frac{\lfloor n}{\lfloor x\,\lfloor n-x-1}p^{x+1}q^{n-x}\right]$$

<u>Define</u>

$$t_x = \frac{\lfloor n}{\lfloor x-1\,\lfloor n-x}p^{x}q^{n-x+1}$$

$$\therefore MD_\mu = 2\sum_{x=\nu}^{n}(t_x-t_{x+1})$$

$$= 2[t_\nu - t_{\nu+1} + t_{\nu+1} - t_{\nu+2} + t_{\nu+2} + \cdots + t_{n-1} - t_n + t_n]$$

$$= 2t_\nu$$

$$= 2\frac{\lfloor n}{\lfloor \nu-1\,\lfloor n-\nu}p^\nu q^{n-\nu+1}$$

$$\Rightarrow MD_\mu = 2\frac{\lfloor n}{\lfloor [np]\,\lfloor n-[np]-1}p^{[np]+1}q^{n-[np]} \quad [\text{Since } [np]+1=\nu]$$

Recursion Relation between the central moments of Binomial Distribution

$$\mu_r = E(X-\mu)^r = E(X-np)^r$$

$$= \sum_{x=0}^{n}(x-np)^r\binom{n}{x}p^x(1-p)^{n-x}$$

Differentiate μ_r w.r.t p

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} r(x-np)^{r-1}(-n)\binom{n}{x}p^x(1-p)^{n-x}$$

$$+ \sum_{x=0}^{n}(x-np)^r\binom{n}{x}xp^{x-1}(1-p)^{n-x}$$

$$+ \sum_{x=0}^{n}(x-np)^r\binom{n}{x}p^x(n-x)(1-p)^{n-x-1}(-1)$$

$$= -nr\mu_{r-1} + \sum_{x=0}^{n}(x-np)^r\binom{n}{x}p^x q^{n-x}\left(\frac{x}{p}-\frac{n-x}{q}\right)$$

$$= -nr\mu_{r-1} + \sum_{x=0}^{n}(x-np)^r\binom{n}{x}p^x q^{n-x}\left(\frac{x-np}{pq}\right)$$

$$\frac{d\mu_r}{dp} = -nr\mu_{r-1} + \frac{1}{pq}\mu_{r+1}$$

$$\Rightarrow \mu_{r+1} = pq\left[\left(\frac{d\mu_r}{dp}\right) + nr\mu_{r-1}\right]$$

Put $r = 2$, we have

$$\mu_3 = pq\left[\frac{d}{dp}\mu_2 + 2n\,\mu_1\right]$$

$$= pq\left[\frac{d}{dp}\,np(1-p)\right]$$

$$= pq\left[n - 2np\right]$$

$$= npq(1-2p)$$

Put $r = 3$, we have

$$\mu_4 = pq\left[\frac{d}{dp}\mu_3 + 3n\,\mu_2\right]$$

$$= pq\left[\frac{d}{dp}\{np(1-p)(1-2p)\} + 3n^2pq\right]$$

$$= pq\left[\frac{d}{dp}(np - 3np^2 + 2np^3) + 3n^2pq\right]$$

$$= pq\left[n - 6np + 6np^2 + 3n^2pq\right]$$

$$= npq\left[1 - 6p + 6p^2 + 3npq\right]$$

$$= npq\left[1 - 6pq + 3npq\right]$$

Probability Generating Function

$$P_X(t) = E(t^X)$$

$$= \sum_{x=0}^{\infty} t^x\, P[X = x]$$

$$= \sum_{x=0}^{\infty} t^x \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{\infty} \binom{n}{x} (pt)^x q^{n-x}$$

$$= (pt+q)^n$$

$$P_X(t)\Big|_{t=0} = q^n = 0!\,f(0)$$

$$\frac{d}{dt}P_X(t)\Big|_{t=0} = np\,(pt+q)^{n-1}\Big|_{t=0} = npq^{n-1} = 1!\,f(1)$$

$$\frac{d^2}{dt^2}P_X(t)\Big|_{t=0} = n(n-1)p^2\,(pt+q)^{n-2}\Big|_{t=0}$$

$$= n(n-1)p^2q^{n-2}$$

$$= 2!\,f(2)$$

Moment Generating Function :— [gives raw moments]

$$M_x(t) = E(e^{tx})$$
$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}$$
$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x}$$
$$= (pe^t + q)^n$$

$$M_x(t)\big|_{t=0} = 1$$

$$\frac{d}{dt} M_x(t)\big|_{t=0} = n(pe^t+q)^{n-1} pe^t\big|_{t=0}$$
$$= np(p+q)^{n-1}$$
$$= np = \mu_1' = E(x)$$

$$\frac{d^2}{dt^2} M_x(t)\big|_{t=0} = n(n-1)(pe^t+q)^{n-2}(pe^t)^2 + n(pe^t+q)^{n-1} pe^t\big|_{t=0}$$
$$= n(n-1)p^2 + np$$
$$= n^2p^2 - np^2 + np$$
$$= n^2p^2 + np(1-p)$$
$$= n^2p^2 + npq = \mu_2' = E(x^2)$$

Now $V(x) = \mu_2' - (\mu_1')^2 = n^2p^2 + npq - n^2p^2$
$$= npq$$

Now $$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{n^2p^2q^2(1-2p)^2}{n^2p^2q^2} = (1-2p)^2$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{npq(1-6pq+3npq)}{n^2p^2q^2} = \frac{1-6pq+3npq}{npq}$$
$$= 3 + \frac{1-6pq}{npq}$$

∴ Skewness
$$\gamma_1 = \sqrt{\beta_1} = \sqrt{(1-2p)^2} = 1-2p$$

Kurtosis
$$\gamma_2 = \beta_2 - 3 = 3 + \frac{1-6pq}{npq} - 3 = \frac{1-6pq}{npq}$$

Factorial Moment :-

Factorial moment of order 'r' about zero is

$$\mu'_{[r]}(0) = E[x(x-1)(x-2)\cdots(x-r+1)]$$

$$= \sum_{x=0}^{n} [x(x-1)(x-2)\cdots(x-r+1)] \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} [x(x-1)(x-2)\cdots(x-r+1)] \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= n(n-1)(n-2)\cdots(n-r+1)\, p^r \sum_{x=r}^{n} \frac{(n-r)!}{(x-r)!(\overline{n-r}-\overline{x-r})} p^{n-r} q^{\overline{n-r}-\overline{x-r}}$$

$$= n(n-1)(n-2)\cdots(n-r+1)\, p^r$$

For $r = 1, 2, 3, 4$ we have

$$\mu'_{[1]}(0) = np$$

$$\mu'_{[2]}(0) = n(n-1)p^2$$

$$\mu'_{[3]}(0) = n(n-1)(n-2)p^3$$

$$\mu'_{[4]}(0) = n(n-1)(n-2)(n-3)p^4$$

$$\therefore \mu'_1(0) = E(x) = \mu'_{[1]}(0) = np$$

$$\mu'_2(0) = E(x^2) = E[x(x-1)+x]$$

$$= E[x(x-1)] + E(x)$$

$$= \mu'_{[2]}(0) + \mu'_{[1]}(0)$$

$$= n(n-1)p^2 + np$$

$$\mu'_3(0) = E(x^3) = E[x(x-1)(x-2) + 3x(x-1) + x]$$

$$= \mu'_{[3]}(0) + 3\mu'_{[2]}(0) + \mu'_{[1]}(0)$$

$$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

$$\mu_4'(0) = E(x^4)$$

$$= E[x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x]$$

$$= \mu_{[4]}'(0) + 6\mu_{[3]}'(0) + 7\mu_{[2]}'(0) + \mu_{[1]}'(0)$$

$$= n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

So, $\mu = \text{mean} = \mu_1'(0) = np$

Now the central moments are

$$\mu_2 = \mu_2' - \mu_1'^2 = n(n-1)p^2 + np - (np)^2.$$

$$= np - np^2 = np(1-p) = npq$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np - 3np[n(n-1)p^2 + np]$$
$$+ 2n^3p^3$$

$$= (n^3 - 3n^2 + 2n)p^3 + 3n^2p^2 - 3np^2 + np - 3np[n^2p^2 - np^2 + np]$$
$$+ 2n^3p^3$$

$$= n^3p^3 - 3n^2p^3 + 2np^3 + 3n^2p^2 - 3np^2 + np - 3n^3p^3 + 3n^2p^3$$
$$- 3n^2p^2 + 2n^3p^3$$

$$= 2np^3 - 3np^2 + np$$

$$= np[2p^2 - 3p + 1]$$

$$= np[2p^2 - 2p - p + 1]$$

$$= np[-2p(1-p) + (1-p)]$$

$$= np(1-p)(1-2p)$$

$$= npq(1-2p)$$

and $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$$= n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$
$$- 4np\,[n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np]$$
$$+ 6n^2p^2\,[n(n-1)p^2 + np] - 3n^4p^4$$

$$= (n^4 - 6n^3 + 11n^2 - 6n)p^4 + (6n^3 - 18n^2 + 12n)p^3 + 7n^2p^2 - 7np^2 + np$$
$$- 4np\,[(n^3 - 3n^2 + 2n)p^3 + 3n^2p^2 - 3np^2 + np]$$
$$+ 6n^2p^2\,[n^2p^2 - np^2 + np] - 3n^4p^4$$

$$= n^4p^4 - 6n^3p^4 + 11n^2p^4 - 6np^4 + 6n^3p^3 - 18n^2p^3 + 12np^3$$
$$+ 7n^2p^2 - 7np^2 + np - 4n^4p^4 + 12n^3p^4 - 8n^2p^4 - 12n^3p^3$$
$$+ 12n^2p^3 - 4n^2p^2 + 6n^4p^4 - 6n^3p^4 + 6n^3p^3 - 3n^4p^4$$

$$= [n^4p^4 - 4n^4p^4 + 6n^4p^4 - 3n^4p^4] + [-6n^3p^4 + 12n^3p^4 - 6n^3p^4]$$
$$+ [11n^2p^4 - 8n^2p^4] - 6np^4 + [6n^3p^3 - 12n^3p^3 + 6n^3p^3]$$
$$+ [-18n^2p^3 + 12n^2p^3] + 12np^3 + [7n^2p^2 - 4n^2p^4]$$
$$- 7np^2 + np$$

$$= 3n^2p^4 - 6np^4 - 6n^2p^3 + 12np^3 + 3n^2p^2 - 7np^2 + np$$

$$= (3n^2p^2 - 6n^2p^3 + 3n^2p^4) + [np - 7np^2 - 12np^3 - 6np^4]$$

$$= 3n^2p^2(1 - 2p + p^2) + [(np - np^2) - 6np^2(1 - 2p + p^2)]$$

$$= 3n^2p^2(1-p)^2 + [np(1-p) - 6np^2(1-p)^2]$$

$$= 3n^2p^2q^2 + (npq - 6np^2q^2)$$

$$= 3n^2p^2q^2 + npq(1 - 6pq)$$

: Poisson Distribution :

$$\lim_{\substack{n\to\infty \\ p\to 0 \\ np=\lambda}} f(x) = \lim_{\substack{n\to\infty \\ p\to 0 \\ np=\lambda}} \frac{\lfloor n}{\lfloor x \lfloor n-x} p^x q^{n-x}$$

$$= \frac{1}{\lfloor x} \lim_{\substack{n\to\infty \\ p\to 0 \\ np=\lambda}} n(n-1)(n-2)\cdots(n-x+1)\, p^x q^{n-x}$$

$$= \frac{1}{\lfloor x} \lim_{\substack{n\to\infty \\ p\to 0 \\ np=\lambda}} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)(np)^x \left(1-\frac{np}{n}\right)^{n-x}$$

$$= \frac{1}{\lfloor x} \lim_{n\to\infty} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\lambda^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{1}{\lfloor x} \lim_{n\to\infty} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\lambda^x \left(1-\frac{\lambda}{n}\right)^{n} \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{\lfloor x} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{\lambda^x}{\lfloor x} e^{-\lambda} = \frac{e^{-\lambda}\lambda^x}{x!}$$

$\therefore$ A random variable is said to follow a Poisson distⁿ with parameter λ iff. its p.m.f can be given by

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad ; \quad x = 0, 1, 2, \ldots$$
$$= 0 \quad ; \quad \text{o.w}$$

Expectation :-

$$E(x) = \sum_{x=0}^{\infty} x f(x)$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \lambda \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} = \lambda \cdot 1 \qquad \left[\because \sum_{x=0}^{\infty} f(x) = 1\right]$$
$$= \lambda$$

Variance :-

$$V(x) = E(x^2) - [E(x)]^2$$
$$= E[x(x-1)] + E(x) - [E(x)]^2$$

Now $E[x(x-1)] = \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!}$

$$= \lambda^2 \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} = \lambda^2 \cdot 1 = \lambda^2$$

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$\therefore \; V(x) = \lambda^2 + \lambda - \lambda^2 = \lambda$

Mode of a Poisson Distribution

Let h be the mode of poisson distⁿ. Therefore h must satisfy the following two inequalities:

i) $f(h) \geq f(h+1)$

ii) $f(h) \geq f(h-1)$

Now, $$\frac{f(h)}{f(h+1)} = \frac{e^{-\lambda}\lambda^{h}/h!}{e^{-\lambda}\lambda^{h+1}/(h+1)!} = \frac{h+1}{\lambda}$$

Now from (i) we get

$$\frac{h+1}{\lambda} \geq 1 \Rightarrow h \geq \lambda - 1 \quad \text{—— (iii)}$$

Now $$\frac{f(h)}{f(h-1)} = \frac{e^{-\lambda}\lambda^{h}/h!}{e^{-\lambda}\lambda^{h-1}/(h-1)!} = \frac{\lambda}{h}$$

So from (ii) we get

$$\frac{\lambda}{h} \geq 1 \Rightarrow h \leq \lambda \quad \text{—— (iv)}$$

From (iii) & (iv) we get

$$\lambda - 1 \leq h \leq \lambda$$

Case-I:

λ is an integer, so is $(\lambda - 1)$. Therefore no integer ~~ca~~ ~~bet~~ remain in between λ and $(\lambda-1)$. But h is an integer. So in this case the distⁿ has two modes λ & $(\lambda-1)$. And hence the distⁿ in this case is bimodal.

Case-II:

λ is not an integer, so is $(\lambda-1)$. There is ~~no~~ only one integer in between λ & $(\lambda-1)$ i.e. $[\lambda]$. So in this case mode of the distⁿ will be $[\lambda]$. Hence the distⁿ in this case is unimodal.

Mean Deviation about mean :-

MD_μ (mean deviation about mean) $= E|X-\mu|$

$$= \sum_{x=0}^{\infty} |x-\lambda| f(x)$$

$$= \sum_{x=0}^{\infty} |x-\lambda| e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= 2\sum_{x=k+1}^{\infty} (x-\lambda) e^{-\lambda} \frac{\lambda^x}{x!} \text{, where } k=[\lambda]$$

$$= 2e^{-\lambda} \sum_{x=k+1}^{\infty} \left\{ \frac{\lambda^x}{(x-1)!} - \frac{\lambda^{x+1}}{x!} \right\}$$

$$= 2e^{-\lambda} \sum_{x=k+1}^{\infty} (\vartheta_x - \vartheta_{x+1}) , \quad \vartheta_x = \frac{\lambda^x}{(x-1)!}$$

$$= 2e^{-\lambda} \vartheta_{k+1}$$

$$= 2e^{-\lambda} \frac{\lambda^{k+1}}{k!}$$

$$= 2\lambda \frac{e^{-\lambda} \lambda^k}{k!} .$$

Recursion Relation between Central moment of Poisson Distribution :-

$$\mu_r = E(x-\mu)^r$$
$$= E(x-\lambda)^r = \sum_{x=0}^{\infty}(x-\lambda)^r \frac{e^{-\lambda}\lambda^x}{x!}$$

Now $$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty}(-r)(x-\lambda)^{r-1}\frac{e^{-\lambda}\lambda^x}{x!} - \sum_{x=0}^{\infty}(x-\lambda)^r\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty}(x-\lambda)^r\frac{e^{-\lambda}x\lambda^{x-1}}{x!}$$

$$= -r\mu_{r-1} + \sum_{x=0}^{\infty}(x-\lambda)^r\frac{e^{-\lambda}\lambda^{x-1}}{x!}(x-\lambda)$$

$$= -r\mu_{r-1} + \frac{1}{\lambda}\sum_{x=0}^{\infty}(x-\lambda)^{r+1}\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= -r\mu_{r-1} + \frac{1}{\lambda}\mu_{r+1}$$

$$\Rightarrow \mu_{r+1} = \lambda\left[\frac{d}{d\lambda}\mu_r + r\mu_{r-1}\right]$$

Now put $r=2$

$$\mu_3 = \lambda\left[\frac{d}{d\lambda}\mu_2 + 2\mu_1\right]$$
$$= \lambda\left[\frac{d}{d\lambda}(\lambda) + 0\right] = \lambda$$

Put $r=3$

$$\mu_4 = \lambda\left[\frac{d}{d\lambda}\mu_3 + 3\mu_2\right]$$
$$= \lambda\left[\frac{d}{d\lambda}(\lambda) + 3\lambda\right] = \lambda[1+3\lambda]$$

$$\therefore \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda} \qquad \left[\text{Since } \begin{matrix}\mu_1 = 0\\ \mu_2 = \lambda\end{matrix}\right]$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\lambda[1+3\lambda]}{\lambda^2} = \frac{1+3\lambda}{\lambda} = \frac{1}{\lambda}+3$$

$\therefore$ Skewness:

$$\gamma_1 = \sqrt{\beta_1} = \sqrt{\frac{1}{\lambda}}$$

Kurtosis

$$\gamma_2 = \beta_2 - 3 = \frac{1}{\lambda}+3-3 = \frac{1}{\lambda}$$

Probability Generating Function

$P_X(t) = E[t^X]$

Now $E(t^x) = \sum_{x=0}^{\infty} t^x f(x)$

$$= \sum_{x=0}^{\infty} t^x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda t)^x}{x!} = e^{-\lambda} e^{\lambda t} = e^{\lambda(t-1)}$$

$$P_X(t)\big|_{t=0} = e^{-\lambda} = 0!\, f(0)$$

$$\frac{d P_X(t)}{dt}\Big|_{t=0} = \lambda e^{\lambda(t-1)}\Big|_{t=0} = \lambda e^{-\lambda} = 1!\, f(1)$$

$$\frac{d^2 P_X(t)}{dt}\Big|_{t=0} = \lambda^2 e^{\lambda(t-1)}\Big|_{t=0} = \lambda^2 e^{-\lambda} = 2!\, f(2)$$

Moment Generating Function

$M_X(t) = E[e^{Xt}]$

$$E[e^{Xt}] = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)} = e^{-\lambda(1-e^t)}$$

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = \lambda = \mu_1'$$

$$\frac{d^2}{dt} M_X(t)\Big|_{t=0} = \left[\lambda e^t e^{-\lambda(1-e^t)} + \lambda^2 e^{2t} e^{-\lambda(1-e^t)}\right]\Big|_{t=0}$$

$$= \lambda + \lambda^2 = \mu_2'$$

$$\therefore V(x) = \mu_2' - (\mu_1')^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$