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NAAC ACCREDITED 'A' GRADE

Topic: Sequene of Real numbers

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Academic Script

1. Introduction

Friends, we are well familiar with the idea of summing finitely many numbers or quantities but today we are going to learn the concept of infinite sums which is needed quite often in mathematics. The idea of infinite sums is related to the concept of convergent and divergent sequences and so we shall begin with the formal definition of sequence and its limit.

2. Sequence

An infinite sequence of real numbers is a function f with domain as set of natural numbers $\mathbb{N}$ and codomain as set of real numbers$\mathbb{R}$.

Although by definition, sequence is a function f on the set $\mathbb{N}$, usually it is written as $\{x_n\}_{n=1}^{\infty}$ or as

$x_1, x_2, x_3, \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Where x_n is the value of this function at the point $n \in \mathbb{N}$, i.e., $x_n = f(n)$. Thus intuitively, a sequence is simply a list of numbers. Our major interest here is whether a given sequence of numbers $\{x_n\}_{n=1}^{\infty}$ approaches a fixed number as n approaches infinity. Let us make this point more precise by defining the limit of a sequence.

3. Limit of a sequence

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers and l be any real number. We say that the sequence $\{x_n\}_{n=1}^{\infty}$ has limit l or it converges to l, and we write,

$lim_{n\to\infty} x_n = l$ or $x_n \to l$ as $n \to \infty$, If for every $\epsilon > 0$, there exists a positive integer n_0 such that $|x_n - l| < \epsilon$ for all $n \geq n_0$.

From the definition, we understand that a sequence $\{x_n\}_{n=1}^{\infty}$ has limit l if x_n is sufficiently close to l for all n, sufficiently large. Also it is a very simple task to show that if the limit of a sequence exists, then it is unique. If a sequence has the limit then it is called a ***convergent sequence*** otherwise it is termed as a ***divergent sequence***. Note that the simplest example of a convergent sequence is a constant sequence, i.e., $x_n = \alpha$ for all$n$. It is obvious that whether a given sequence $\{x_n\}_{n=1}^{\infty}$ is convergent or divergent depends purely on the behavior of x_n for large values of n and so the first thousand or the first billion terms of the sequence are irrelevant when we are discussing issues related to the convergence or divergence of the sequences.

Properties of Limit

Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences. Then

1. $\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n$.
2. $\lim_{n\to\infty}(x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$.
3. $\lim_{n\to\infty}(x_n/y_n) = \lim_{n\to\infty} x_n / \lim_{n\to\infty} y_n$, provided $\lim_{n\to\infty} y_n \neq 0$.
4. $\lim_{n\to\infty}(\alpha\, x_n) = \alpha \lim\limits_{n\to\infty} x_n \; for\ every\ real\ \alpha.$

We now discuss few theorems which are helpful in understanding the concept of convergent sequences. For this we require some definitions.

Definition: Bounded sequence

A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is said to be bounded if there exists a real number M such that

$|x_n| \leq M \; for\ all\, n.$

Definition: Increasing/decreasing sequences

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then it is said to be increasing if $x_n \le x_{n+1}$ for all n. On the other hand if $x_n \ge x_{n+1}$ for all n then the sequence is said to be decreasing.

A sequence which is either increasing or decreasing is also called a **monotonic sequence.**

Theorem: Every convergent sequence is bounded.

Theorem: Every monotonic bounded sequence is convergent.

Theorem: If $x_n \to l$ as $n \to \infty$, then for every $\epsilon > 0$, there exists a positive integer n_0 such that $|x_n - x_m| < \epsilon$ for all $n, m \ge n_0$.

Let us now take a simple example.

Example: Find the limit of the sequence

$$1, \frac{1}{4}, \frac{1}{9}, \dots, \frac{1}{n^2}, \dots\dots\dots$$

Solution: Let $x_n = \frac{1}{n^2}$. clearly this sequence $\{x_n\}$ is decreasing and bounded and so it is convergent. We see that as n gets larger and larger x_n gets closer and closer to 0. Hence, we expect the limit of this sequence to be0. To prove this, given a positive number ϵ, we want to show that there exists a positive integer n_0 such that

$\frac{1}{n^2} < \epsilon$ For all $n \ge n_0$.

But it is easy to see here that if we choose n_0 to be any positive integer greater than$1/\sqrt{\epsilon}$, then our requirement is fulfilled and so we have

$$\lim_{n\to\infty} \frac{1}{n^2} = 0.$$

<u>About divergent sequences</u>

Consider the sequences $x_n = (-1)^n$ and $y_n = n^2$.

It is clear that none of these two sequences approach any finite number or limit no matter how far we go down the sequence, and so these two sequences are divergent sequences. Although $x_n = (-1)^n$ and $y_n = n^2$ both are divergent sequences.

there is a difference in the way they diverge. Note that the sequence $\{y_n\}$ does not converge to a finite number because its terms are approaching (positive) infinity as n tends to infinity whereas the sequence $\{x_n\}$ does not converge because its terms are bouncing around indefinitely and so cannot settle to a specific value. Such divergent sequences are also known as *oscillatory* sequences.

<u>Sequences with limit as $+\infty$ or $-\infty$</u>

We have given a precise meaning of the notation

$$\lim_{n\to\infty} x_n = l,$$

When l is a finite number but sometimes it is equally important to understand the meaning of

$\lim_{n\to\infty} x_n = +\infty$ Or $\lim_{n\to\infty} x_n = -\infty$.

Definition:

For a sequence $\{x_n\}$, if for every $M > 0$ there exists a positive integer n_0 such that

$$x_n > M \text{ for all } n \geq n_0,$$

Then we say that x_n tends to (positive) infinity and write,

$$\lim_{n\to\infty} x_n = +\infty.$$

On the other hand if for every $M < 0$ there exists a positive integer n_0 such that

$$x_n < M \text{ for all } n \geq n_0,$$

Then we say that x_n tends to (negative) infinity and write,

$$\lim_{n\to\infty} x_n = -\infty.$$

Note that if $\{x_n\}$ is a sequence of non-zero numbers which tends to positive or negative infinity, then

$$\lim_{n\to\infty} \frac{1}{x_n} = 0.$$

Theorem: Let r be any real number and $p > 0$.

1) $\lim_{n\to\infty} n^p = \infty$.
2) If $|r| > 1$, then $\lim_{n\to\infty} |r|^n = \infty$.
3) If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$.
4) $\lim_{n\to\infty} p^{1/n} = 1$.
5) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.
6) $\lim_{n\to\infty} \frac{n^r}{(1+p)^n} = 0$.

Since we are often encountered with finding the limit of quotient of polynomials $\frac{p(n)}{q(n)}$, we take an example to understand it.

Example: Determine $\lim_{n\to\infty} \frac{2n^3+n-1}{n^3+n^2+6}$.

Solution: Note that

$$\frac{2n^3+n-1}{n^3+n^2+6} = \frac{n^3\left(2+\frac{1}{n^2}-1/n^3\right)}{n^3\left(1+\frac{1}{n}+6/n^3\right)} = \frac{\left(2+\frac{1}{n^2}-1/n^3\right)}{\left(1+\frac{1}{n}+6/n^3\right)}$$

Using the properties of limit, it follows that the required limit is

$$\frac{\lim\limits_{n\to\infty}\left(2+\frac{1}{n^2}-1/n^3\right)}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}+6/n^3\right)} = \frac{2}{1} = 2.$$

The technique used in this example can be used to derive a general result about the limit of a quotient of any two polynomials.

Theorem: If $p(x)$ and $q(x)$ are polynomials with leading coefficients as $l > 0$ and $m > 0$ respectively then

a) $\lim_{n\to\infty} \frac{p(n)}{q(n)} = \frac{l}{m}$ if $p(x)$ and $q(x)$ are of same degree

b) $\lim_{n\to\infty} \frac{p(n)}{q(n)} = 0$ if degree of $q(x)$ exceeds that of $p(x)$

c) $\lim_{n\to\infty} \frac{p(n)}{q(n)} = \infty$ if degree of $p(x)$ exceeds that of $q(x)$.

In mathematics as well as economics the sequence $\left(1+\frac{1}{n}\right)^n$ has a distinguished importance and hence we need to know its behavior as n tends to infinity. Since this sequence is increasing and bounded, it is convergent and its limit is defined to be the number e. It is an irrational number whose approximate value is 2.71828. In mathematics there are many equivalent definitions of e, but here we would like to see an economic interpretation of this number e. Note that $\left(1+\frac{1}{n}\right)^n$ is the year-end amount to the principal amount of 1 unit, if it is assumed to grow at 100% per annum and if the interest is compounded n times in a year at regular intervals. With this interpretation, e is simply the year-end amount to the principal amount of 1 unit, if it is assumed to grow at 100% per annum and if the interest is compounded continuously?

In economics, we often deal with exponential, logarithmic and polynomial functions and on certain occasions it is important to compare the growth of these functions with each other. The following result highlights this matter.

Theorem: Let t be any positive number. Then

1) $\lim_{n\to\infty} \frac{\log n}{(1+t)^n} = 0.$

2) $\lim_{n\to\infty} \frac{p(n)}{(1+t)^n} = 0$ where $p(n)$ is a polynomial in n.

3) $\lim_{n\to\infty} \frac{\log n}{p(n)} = 0$ where $p(n)$ is a polynomial in n.

It can be inferred from these three results that the growth of the exponential function is much faster as compared to the growth of logarithmic or polynomial functions. Further, the growth of the