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NAAC ACCREDITED 'A' GRADE

Topic: Infinite Series

Course Title:

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Series

* Must read the introduction of your book *

Definition of an infinite series

Let us suppose that we have a sequence $\{u_n\}_n, n \in \mathbb{N}$. The ~~infinite~~ infinite sum or expression like $u_1 + u_2 + u_3 + \cdots \infty$ is called an infinite series or simply a series. The numbers $u_1, u_2, u_3, \ldots$ are called the terms of the series.

Note:– 1) A series is always meant for infinite series

2) The above series may also be denoted by $\sum_{n=1}^{\infty} u_n$ or by $\sum_{1}^{\infty} u_n$ or by $\sum u_n$

3) In place of $\sum u_n$, we may denote a series by $\sum x_n$, $\sum a_n$, $\sum v_n$ or by any variable names.

4) Terms of a series ~~are~~ or a sequence may be positive, negative or mixed.

Partial sum

The sum of first n terms of a series $\sum u_n$ is called the n-th partial sum of the series and is denoted by s_n. Hence $s_n = u_1 + u_2 + \cdots + u_n$

Thus $s_1 = u_1$, $s_2 = u_1 + u_2$, $s_3 = u_1 + u_2 + u_3$

$\cdots$ $s_n = u_1 + u_2 + \cdots + u_n$

$\therefore$ We get a sequence $\{s_n\}$, which is called sequence of partial sums.

Note that $\lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} u_n$

So, if $\lim_{n\to\infty} s_n$ exists and $\lim_{n\to\infty} s_n = s$ then

we have $\lim_{n\to\infty} s_n = s = \sum_{n=1}^{\infty} u_n$

So, if $\lim_{n\to\infty} s_n$ exists finitely, say s, then we say that the series is convergent and converges to s (s is also called the sum of the series). On the other hand if $\lim_{n\to\infty} s_n$ does not exists finitely, we say that the series is divergent or does not converge.

* Illustration - 1:-

Consider the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

Now let $\{s_n\}$ be the sequence of partial sums

Then $s_1 = \frac{1}{2}$

$s_2 = \frac{1}{2} + \frac{1}{4}$

$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

$s_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

Notes that $s_1 = \frac{1}{2} = 1 - \frac{1}{2}$

$s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{4} = 1 - \frac{1}{2^2}$

$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1 - \frac{1}{2^3}$

$s_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1 - \frac{1}{2^4}$

similarly $s_n = 1 - \frac{1}{2^n}$

Now $\lim_{n\to\infty} s_n = \lim_{n\to\infty} (1 - \frac{1}{2^n}) = 1$

$\therefore$ The sum of the series is 1, i.e. the series is convergent and converges to 1.

* Illustration-2:-

Consider the series :- $1+2+3+4+\cdots$

$\therefore$ The nth partial sum

$$s_n = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Now $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \frac{n(n+1)}{2} = \infty$

Thus the given series is not convergent i.e. divergent

Note that we convert a series to a sequence by the sequence of partial sums.

* Necessary condition for the Convergence of a series

Theorem :- If a series $\sum u_n$ converges, then $\lim\limits_{n\to\infty} u_n = 0$

[Proof is not needed]

So, for a series if $\lim\limits_{n\to\infty} u_n \neq 0$ then the series $\sum u_n$ will never be convergent.

$\Rightarrow$ We have that the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ is convergent i.e. $\sum\limits_{n=1}^{\infty} \frac{1}{2^n}$ is convergent

Now $\lim\limits_{n\to\infty} \frac{1}{2^n} = 0$

$\Rightarrow$ Consider the series $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots$ i.e. the series $\sum\limits_{n=1}^{\infty} \frac{n}{n+1}$. Now $u_n = \frac{n}{n+1}$.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1 \neq 0$$

$\therefore$ The series $\sum u_n$ i.e. $\sum\limits_{n=1}^{\infty} \frac{n}{n+1}$ is not convergent

* ~~Remember~~

The converse of that theorem is not true

~~Test~~ i.e. if a series $\sum_{n=1}^{\infty} u_n$ is convergent then $\lim_{n\to\infty} u_n = 0$, but if $\lim_{n\to\infty} u_n = 0$, it does not imply that the $\sum_{n=1}^{\infty} u_n$ is convergent.

For an example consider the series
$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

This series is not convergent (Proof is given in the book ~~s~~on page numbers 27) but

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$\therefore$ $\lim_{n\to\infty} u_n = 0$ does not imply the convergence of the series $\sum u_n$

but if $\lim_{n\to\infty} u_n \neq 0$ then the series is divergent.

* A series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$

* A geometric series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$ ($a \neq 0$; a & r are constants) is convergent and converges to $\frac{a}{1-r}$ if $|r| < 1$ and divergent if $|r| \geq 1$

A series $\sum u_n$ converges, if the sequence of corresponding partial sums $\{s_n\}$ converges and vice versa. Thus the convergence criteria of $\sum u_n$ and $\{s_n\}$ are equivalent.

~~A necce~~

Cauchy's General principle for Convergence with Applications :-

A necessary and sufficient condition for the convergence of $\{s_n\}$ is that corresponding to any arbitrary $\varepsilon(>0)$, there exists a positive integer m such that $|s_{n+p} - s_n| < \varepsilon$ $\forall\ n \geq m$ and for all positive integer p.

How $|s_{n+p} - s_n| < \varepsilon$

$$\Rightarrow \left|(u_1+u_2+\cdots+u_{n+p}) - (u_1+u_2+\cdots+u_n)\right| < \varepsilon$$

$$\Rightarrow |u_{n+1}+u_{n+2}+\cdots+u_{n+p}| < \varepsilon$$

$\forall\ n \geq m$ and for all positive integer p.

Application

We will prove the series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is not convergent by Cauchy's principle for convergence.

If possible let the series $\sum u_n$ converges. Choose $\varepsilon = 0.4$, so there exists a positive integer m, such that

$$|u_{n+1}+u_{n+2}+u_{n+3}+\cdots+u_{n+p}| < 0.4$$

$\forall\ n \geq m$ & for all $p \in \mathbb{N}$

$$\Rightarrow \left|\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+p}\right| < 0.4 \quad \forall\ n \geq m\ \&\ \forall\ p \in \mathbb{N}$$

In particular we take $n = m$ & $p = m$, Thus we obtain

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} = \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{m+m}$$

$$> \frac{1}{2m} + \frac{1}{2m} + \cdots + \frac{1}{2m}$$

(m times)

$$= m \cdot \frac{1}{2m} = \frac{1}{2}$$

But $\frac{1}{2} = 0.5 > 0.4 = \varepsilon$, which is a contradiction

$$\therefore \left| \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} \right| \not< 0.4$$

$$\forall n \geq m \ \& \ \forall p \in \mathbb{N}$$

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by Cauchy general principle for convergence

* A series of positive terms means every term of the series is positive.

There are some tests to examine if a series is convergent or not.

For a series of positive terms there are 4 tests we have to learn.

i) Comparison test

ii) D'Alembert's Ratio test

iii) Cauchy's n-th Root test

& iv) Raabe's test.

i) Comparison test

statements

a) If $\sum u_n$ and $\sum v_n$ be two series of positive terms, such that $u_n \le v_n$ for all $n \geq m$ (m is a positive integer) ~~then~~ and the series $\sum v_n$ is convergent, then the series $\sum u_n$ is also convergent.

b) If $\sum u_n$ and $\sum v_n$ be two series of positive terms, such that $u_n \geq v_n$, for all $n \geq m$ (m is a positive integer) and the series $\sum v_n$ is divergent then the series $\sum u_n$ is also divergent.

Illustration :- We know $\frac{1}{2^n+1} < \frac{1}{2^n}$ for all $n \geq 1$

and $\sum \frac{1}{2^n}$ is convergent hence by statement a)

$\sum \frac{1}{2^n+1}$ is also convergent.

Illustration :- We know that $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$ $\forall\, n \geq 1$

and $\sum \frac{1}{n}$ is divergent then by the statement b)

$\sum \frac{1}{\sqrt{n}}$ is also divergent

~~that~~ We can use this similar argument to prove that the series $\sum \frac{1}{n^p}$ (where $p \le 1$) is divergent as $\frac{1}{n^p} \geq \frac{1}{n}$ (if $p \le 1$) for all $n \geq 1$. Such as $\sum \frac{1}{n^{0.2}}$ is a divergent series.

So, here we compare two series where one of that two series is know. We can deduct the nature of the other series.

there is another form of comparison test which is limit form.

Limit form of comparison test says

if Σu_n and Σv_n are two series of positive terms, such that $\lim_{n \to \infty} \left(\frac{u_n}{v_n}\right) = l$, where l is a non-zero (i.e. $l \neq 0$) finite number, then both the series converges or diverge together.

$\Rightarrow$ For example ~~for~~ take the two series as before i.e. $\Sigma \frac{1}{2^n+1}$ & $\Sigma \frac{1}{2^n}$

Now

$$\lim_{n \to \infty} \left(\frac{1}{2^n}\right) \Big/ \left(\frac{1}{2^n+1}\right) = \lim_{n \to \infty} \frac{2^n+1}{2^n} = \lim_{n \to \infty} \left(1 + \frac{1}{2^n}\right) = 1 \neq 0$$

$\therefore$ $\Sigma \frac{1}{2^n+1}$ & $\Sigma \frac{1}{2^n}$ converge or diverge together i.e. if on of the series is convergent (or divergent) then the other one is convergent (or divergent).

Now we know that the series $\Sigma \frac{1}{2^n}$ is convergent so, $\Sigma \frac{1}{2^n+1}$ is also convergent.

$\Rightarrow$ Consider the two serieses $\sum \frac{1}{2^n}$ & $\sum \frac{1}{n}$

then $\lim_{n\to\infty}\left(\frac{1}{2^n}\right)/\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{n}{2^n} = 0$

So, we can not compare those two serieses as the limit is not non-zero, So, that l should be non-zero finite number

As you can see $\lim_{n\to\infty}\left(\frac{1}{n}\right)/\left(\frac{1}{2^n}\right) = \infty$

Again we can not compare those two serieses.

Note that For two serieses $\sum \frac{1}{\sqrt{n}}$ & $\sum \frac{1}{n}$

$$\lim_{n\to\infty} \frac{n}{\sqrt{n}} = \lim_{n\to\infty} \sqrt{n} = \infty$$

$$\& \quad \lim_{n\to\infty} \frac{\sqrt{n}}{n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

So, we can not compare those two serieses by the limit form of the comparison test but we have compared those two serieses by the statement b) of the comparison test.

II) D'Alembert's Ratio test :-

Statement :- If $\sum_{n=1}^{\infty} u_n$ be a series of positive terms such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, then the series converges for $l<1$ and diverges for $l>1$. We can not conclude anything if $l=1$.

Illustration :- Consider the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$.

Now $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ [Here $u_n = \frac{2^n}{n!}$]

$= \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \times \frac{n!}{2^n}$

$= \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$

$\therefore$ The series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ is convergent by D'Alembert's Ratio test.

Illustration : Consider the series $\sum_{n=1}^{\infty} \frac{2^n}{n}$

Here $u_n = \frac{2^n}{n}$ then $u_{n+1} = \frac{2^{n+1}}{n+1}$

Now $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2^{n+1}}{n+1} \cdot \frac{n}{2^n}$

$= \lim_{n\to\infty} 2 \cdot \frac{n}{n+1}$

$= \lim_{n\to\infty} 2 \cdot \left(\frac{n+1-1}{n+1}\right)$

$= \lim_{n\to\infty} 2 \cdot \left(1 - \frac{1}{n+1}\right)$

$= 2 > 1$

$\therefore$ The series $\sum_{n=1}^{\infty} \frac{2^n}{n}$ is divergent by D'Alembert's Ratio test.

Illustration :- Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$, Here $u_n = \frac{1}{n}$

$\therefore u_{n+1} = \frac{1}{n+1}$. Now $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n}{n+1}$

$= 1$

$\therefore$ We can not conclude anything about the series by D'Alembert's Ratio test.

iii) Cauchy's n-th Root test :-

Statement :- If $\sum_{n=1}^{\infty} u_n$ be a series of positive terms, such that $\lim_{n\to\infty} \sqrt[n]{u_n} = l$, then the series converges if $l<1$ and diverges for $l>1$ and becomes unconcluded if $l=1$.

Illustration :- Take the series as $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \left[\frac{1}{(1+1/n)^{n^2}}\right]^{1/n} = \lim_{n\to\infty} \frac{1^{1/n}}{(1+1/n)^{n^2\cdot 1/n}}$$

$$= \lim_{n\to\infty} \frac{1}{(1+1/n)^n}$$

$$= \frac{1}{e} < 1$$

So, by Cauchy's n-th Root test, the series $\sum u_n$ is convergent.

Illustration :- Take the series $\sum_{n=1}^{\infty} 1/n$, Here $u_n = 1/n$

Then $\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} (1/n)^{1/n} = \lim_{n\to\infty} \frac{1}{n^{1/n}}$

$$= 1 \quad \left[\text{as } \lim n^{1/n} = 1\right]$$

$\therefore$ We can not conclude anything about that series by Cauchy's n-th Root test (as $l=1$).

iv) Raabe's Test :-

Statement :- If $\sum_{n=1}^{\infty} u_n$ be a series of positive terms, such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$, then the series converges if $l > 1$, diverges if $l < 1$ and becomes ~~unde~~ unconcluded for $l = 1$.

Illustration :- Consider the series

$$\sum_{n=1}^{\infty} u_n = 1 + \frac{1}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7} + \cdots$$

$$\text{Here } u_n = \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-3)}{2\cdot 4\cdot 6\cdot 8\cdots(2n-2)}\cdot\frac{1}{2n-1}$$

$$\therefore u_{n+1} = \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-3)(2n-1)}{2\cdot 4\cdot 6\cdot 8\cdots(2n-2)\cdot 2n}\cdot\frac{1}{2n+1}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{2n\cdot(2n+1)}{(2n-1)^2}$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{2n\cdot(2n+1)}{(2n-1)^2} - 1\right)$$

$$= \lim_{n\to\infty} n\left[\frac{4n^2 + 2n - (4n^2 - 4n + 1)}{(2n-1)^2}\right]$$

$$= \lim_{n\to\infty} n\,\frac{6n-1}{(2n-1)^2}$$

$$= \lim_{n\to\infty} \frac{6 - 1/n}{\frac{(2n-1)^2}{n^2}} = \lim_{n\to\infty} \frac{6 - 1/n}{\left(2 - \frac{1}{n}\right)^2}$$

$$= \frac{6}{4} = \frac{3}{2} > 1$$

$\therefore$ By Raabe's test, the series is convergent.

* Now we are going to consider ~~a~~ series in which terms have alternating signs.

That is the terms have alternately positive and negative or negative and positive signs.

Such a series is called an <u>alternating series</u>.

Thus an alternating series is of the following form:

$u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n+1} u_n + \cdots$

or, $-u_1 + u_2 - u_3 + \cdots + (-1)^n u_n + \cdots$

where $u_1, u_2, u_3, \ldots$ are positive.

It can be expressed as $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ or, $\sum_{n=1}^{\infty} (-1)^n u_n$

For example let the series

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Then this is an alternating series.

Now we have to study the convergence or divergence of alternating series. We have to learn one test which is

<u>Leibnitz's Test</u>

<u>Statement</u>:- If $u_1 - u_2 + u_3 - u_4 + \cdots$ is an alternating series, ($u_i > 0$ for all $i \in \mathbb{N}$)

i) the terms are in a decreasing sequence of positive terms i.e. $u_1 > u_2 > u_3 > u_4 > \cdots$ i.e. $u_{n+1} < u_n \; \forall n \in \mathbb{N}$

and ii) $\lim_{n \to \infty} u_n = 0$, the series $u_1 - u_2 + u_3 - u_4 + \cdots$ converges.

Now the sum of the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$

must satisfy $0 < s \leq u_1$ where $s = \sum_{n=1}^{\infty}(-1)^{n+1}u_n$

Illustration :- Consider the alternating series

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ i.e. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

The terms are in decreasing sequence as

here $u_n = \frac{1}{n}$ and $u_1 > u_2 > u_3 > \cdots$ $\left[\frac{1}{n+1} < \frac{1}{n} \ \forall n \in \mathbb{N}\right]$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n} = 0$

$\therefore$ The alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is

convergent by Leibnitz's test.

$\Rightarrow$ Similarly $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^p}$ is convergent by Leibnitz's test for any $p(>0)$

As $\frac{1}{(n+1)^p} < \frac{1}{n^p} \ \forall n \in \mathbb{N}$ & $\lim_{n\to\infty}\frac{1}{n^p} = 0$

Note that the series (alternating) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is convergent but the series of absolute terms i.e. $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ is not convergent.

Follow the examples