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NAAC ACCREDITED ‘A’ GRADE

Topic: Real Quadratic Form

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Real Quadratic Form

* A homogeneous polynomial of 2nd degree in any number of variables is called quadratic form. If the co-efficients of the variables in a quadratic form are all real then it is called <u>real quadratic form</u>.

The real quadratic form in n variables is $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$, where a_{ij}'s are real and $x_1, x_2, \cdots, x_n$ are variables

You have real quadratic form of maximum three variables in your syllabus

For example, $3x_1^2+5x_2^2+7x_3^2 - 12x_1x_2 - 8x_3x_1 + 7x_2x_3$ is real quadratic form in 3 variables, and $2x_1^2 + 7x_1x_2 + 9x_2^2$ is real quadratic form in 2 variables.

Some times these variables x_1, x_2, x_3 are replaced by x, y, z or any other variable notations.

Note that in any quadratic form a_{ij} is co-efficient of x_ix_j and co-efficient of x_jx_i is a_{ji}. But $x_ix_j = x_jx_i$. So, the coefficient of $x_ix_j (= x_jx_i)$ occures twice in the expression. So, if we take $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) = a_{ji}$ then we got coefficient of x_ix_j is $2a_{ij} (= 2a_{ji})$ and $a_{ij} = a_{ji}$ $\forall\, i,j = 1, 2, \cdots, n$, $i \neq j$ in a quadratic form.

* The Matrix of a Quadratic Form

Let $Q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ be a real quadratic form where $a_{ij} = a_{ji}$.

If we take $A = (a_{ij})_{n \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

and $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, Then the real quadratic form Q can be written as $X^T A X$ i.e. $Q = X^T A X$.

The matrix A, which is a real symmetric matrix, is called the matrix of the quadratic form.

*** Illustration 1:-** We consider the quadratic form

$$Q = ax^2 + 2hxy + by^2.$$

In matrix form it can be written as

$$Q = (x \;\; y) \begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So the matrix of the quadratic form is $\begin{pmatrix} a & h \\ h & b \end{pmatrix}$

*** Illustration 2:-** Here we consider the quadratic form

$$Q = 2x^2 + 5y^2 + 10z^2 + 12yz + 6zx + 4xy$$

In matrix form it can be written as

$$Q = (x \;\; y \;\; z) \begin{pmatrix} 2 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

[Here $x = x_1, y = x_2$, $z = x_3$ & $a_{12} = a_{21} = 2$ and etc.]

$\therefore$ The matrix of the quadratic form is $\begin{pmatrix} 2 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{pmatrix}$

So, you can see that we ~~put~~ devide the co-efficient of xy by 2 and put that in place of a_{12} & a_{21} in the matrix $(a_{ij})_{3\times3}$. Similarly we devide the co-efficient of xz i.e. 6 by 2 and put that in the position of a_{13} & a_{31} and so on.

This is how you create the corresponding matrix of a quadratic form.

Let me give you another example.

Let the quadratic form is

$$2x^2+4y^2+6z^2+5xy+7xz+9yz$$

The the corresponding ~~qua~~ matrix will be

$$\begin{pmatrix} 2 & 5/2 & 7/2 \\ 5/2 & 4 & 9/2 \\ 7/2 & 9/2 & 6 \end{pmatrix}$$

* Different Types of Quadratic Form

A real, quadratic form $Q = X^T AX$, where A is real symmetric matrix, takes up different real values for different X, but $Q=0$ for $X=0$

There are different types of quadratic form such as

1) Positive definite: If a quadratic form $Q>0\ \forall X\neq 0$, then it is called positive definite

Example: $Q = x^2+y^2+z^2$ is a positive definite quadratic form, since $Q>0\ \forall X$ excepte at $X=(0\ 0\ 0)^T$, where $X=(x\ y\ z)^T$

2) Positive semi-definite: If a quadratic form $Q\geq 0$ ~~form~~ for all $X\neq 0$ then it is called positive definite

Example:- $Q = x^2+(y-z)^2$ is a positive semi definite quadratic form, since $x^2\geq 0$ & $(y-z)^2\geq 0$. $Q=0$ for $X=(0\ 1\ 1)^T \neq (0\ 0\ 0)^T$

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3) <u>Negative definite</u>: If a quadratic form $Q<0$ $\forall X \neq 0$ then it is called negative definite.

<u>Example</u>: $Q = -x^2 - y^2 - z^2$ is a negative definite quadratic form since $Q<0, \forall X$ excepte at $X = (0\ 0\ 0)^T$

4) <u>Negative semi-definite</u>: If a quadratic form $Q \leq 0$, $\forall X \neq 0$ then it is called negative semi-definite.

<u>Example</u>: $Q = -x^2 - (y-z)^2$ is a negative semidefinite quadratic form. Here $Q \leq 0$ $\forall X$ since $-x^2 \leq 0$ and $-(y-z)^2 \leq 0$ and $Q=0$ at $X = (0,2,2) \neq (0\ 0\ 0)^T$

5) <u>Indefinite</u>: If a quadratic form $Q>0$ for some $X \neq 0$ and $Q \leq 0$ for some $X \neq 0$ then it is called indefinite quadratic form.

<u>Example</u>: If we take $Q = x^2 + y^2 - z^2$, then $Q>0$ for $X = (1\ 2\ 0)^T$ & $Q<0$ for $X = (1\ 0\ 2)^T$. So, Q is an indefinite quadratic form.

[Remember, Here $X \neq 0$ means $X \neq (0\ 0\ 0)^T$ i.e. X is not zero matrix]

To find the nature of the quadratic form, it is not so easy as the above examples, So we have to go through some different process.

* Nature of the quadratic form can be calculated by the nature of the eigen values of the corresponding matrix. It says that

⇒ A quadratic form $Q(X) = X^T A X$ is

1) Positive definite, iff all the eigen values of A are positive (i.e. >0)

2) Positive semi-definite, iff all the eigen values of A are non-negative (i.e. ≥ 0)

3) Negative definite, iff all the eigen values of A are negative (i.e. <0)

4) Negative semi-definite, iff all the eigen values of A are non-positive (i.e. ≤ 0)

5) In-definite, iff ~~all the~~ A has both the positive and negative eigen values.

(Proof is out of syllabus)

Now you can see that, we need only the nature of the eigen values, not the particular values of them. So, you can calculate the eigen values ~~or~~ or you can reduce the matrix to a diagonal form, ~~and~~ since ~~the~~ ~~or~~ for a diagonal matrix the diagonal elements are the eigen values of ~~of~~ the diagonal matrix, so it will be easy to know their nature. ~~Now since the~~

How will you get the diagonal form?

Just use elementary ~~row~~ congruent operation consist ~~consist~~ of elementary row and column operations, provided that the corresponding elementary matrices are such that each is the transpose of other. Now it says that the nature of the eigen values does not change after elementary congruent operation.

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That is why we can use elementary congruent operation to find the nature of the ~~matrices~~ eigen values of the matrices.

The following are the three types of elementary operation.

i) Interchang of the i-th and j-th row as well as the i-th and j-th column.

ii) Multiplication of the i-th and ~~j-th row~~ as well as j-th column by a non-zero number K.

iii) Addition of K-times the j-th row to the i-th row and of K-times the j-th column to the i-th column.

So, what we do with the ~~column~~ rows, we have to do the same process to the ~~rows~~ columns and we have to make the matrix a diagonal matrix. That is we have to make all the $a_{ij} = 0$ when $i \neq j$.

After we get the ~~dig~~ diagonal matrix, the diagonal elements are the eigen values of the diagonal matrix. So, we will easily get the nature of the eigen values of the original matrix as congruent operation does not change the nature of the eigen values.

Let us see some examples

* Find the nature of the quadratic form

$2x^2 + 3z^2 + 2xy - 6xz + 4yz$

Ans: The matrix A (say) of the quadratic form will be

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & 0 & 2 \\ -3 & 2 & 3 \end{pmatrix}$$

~~Now~~

Method 1:- We shall apply congruence operation on the symmetric matrix A to reduce it to a diagonal matrix

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & 0 & 2 \\ -3 & 2 & 3 \end{pmatrix} \xrightarrow{R_2 - \frac{1}{2}R_1} \begin{pmatrix} 2 & 1 & -3 \\ 0 & -1/2 & 7/2 \\ -3 & 2 & 3 \end{pmatrix}$$

$$\xrightarrow{C_2 - \frac{1}{2}C_1} \begin{pmatrix} 2 & 0 & -3 \\ 0 & -1/2 & 7/2 \\ -3 & 7/2 & 3 \end{pmatrix} \xrightarrow{R_3 + \frac{3}{2}R_1} \begin{pmatrix} 2 & 0 & -3 \\ 0 & -1/2 & 7/2 \\ 0 & 7/2 & -3/2 \end{pmatrix}$$

$$\xrightarrow{C_3 + \frac{3}{2}C_1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1/2 & 7/2 \\ 0 & 7/2 & -3/2 \end{pmatrix} \xrightarrow{R_3 + 7R_2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1/2 & 7/2 \\ 0 & 0 & 23 \end{pmatrix}$$

$$\xrightarrow{C_3 + 7C_2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 23 \end{pmatrix}$$

The eigen values of the resultant diagonal form of the matrix are $2, -1/2$ and 23. So, there are ~~two pos~~ two positive and one negative eigen value.

$\therefore$ The ~~...~~ quadratic form is indefinite.

[**Note that**, what we did with the rows we did the same with the columns. Our main goal is to make the non-diagonal (principle) elements zero. By this process any symmetric matrix can be reduce to diagonal matrix.]

Method 2:- We can directly calculate the eigen values of the matrix A.

The characteristic equation of A is

$$|A - x I_3| = 0$$

$$\text{or, } \begin{vmatrix} 2-x & 1 & -3 \\ 1 & 0-x & 2 \\ -3 & 2 & 3-x \end{vmatrix} = 0$$

$$\text{or, } (2-x)\{(0-x)(3-x)-4\} - 1\{1(3-x)+6\}$$
$$-3\{2-(-3)(0-x)\} = 0$$

$$\text{or, } (2-x)(x^2-3x-4) - (9-x) - 3(2-3x) = 0$$

$$\text{or, } 2x^2 - 6x - 8 - x^3 + 3x^2 + 4x - 9 + x - 6 + 9x = 0$$

$$\text{or, } -x^3 + 5x^2 + 8x - 23 = 0$$

$$\text{or, } x^3 - 5x^2 - 8x + 23 = 0$$

By using Cardon's Method you can solve this cubic equation and you will get

$x = 5.69559, -2.38720, 1.69161$ (approx)

That is there are 2 positive and 1 negative eigen values of A.

$\therefore$ the quadratic form is indefinite.

Note that :- The eigen values of A and the diagonal transform of A are not same but the nature of the eigen values i.e. the number of positive, negative and zero eigen values of both the matrices is same.

Note again that the direct calculation of eigen values of A is difficult in some cases. That is why we use elementary congruent operations to transform the original matrix to a diagonal matrix.

* Find the nature of the quadratic form

$x^2 + 2y^2 + 2z^2 + 2xy + 2xz$

Ans: The matrix A (say) of the given quadratic form is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

We shall apply congruence operation on the symmetric matrix A to reduce it to diagonal form.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow[C_3 - C_1]{C_2 - C_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{C_3 + C_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The eigen values of the resultant matrix are 1, 1, 0.

$\therefore$ There are two positive and one 0 eigen value.

$\therefore$ The quadratic form is positive semi-definite.

> Calculate the eigen values of A directly and try to match the answer i.e. find the nature of the quadratic form by calculating the eigen values of A directly