VIVEKANANDA COLLEGE
THAKURPUKUR
KOLKATA-700063

NAAC ACCREDITED ‘A’ GRADE

Topic: Group Homomorphism

Course Title: Correspondence Theorem

Paper: CC4

Unit: 3

Semester: 2

Name of the Teacher: Jayeeta Saha

Name of the Department: Mathematics

MATHEMATICS

(HONOURS)

SEM-II(CC4,UNIT-3)

GROUP HOMOMORPHISM

(Correspondence Theorem)

Dr. JAYEETA SAHA

ASSISTANT PROFESSOR
DEPARTMENT OF MATHEMATICS,
VIVEKANANDA COLLEGE, THAKURPUKUR

Theorem 0.22. *Correspondence Theorem:*
Let $f : G \to G_1$ *be an epimorpism of groups and* $\mathcal{H} = \{H : H$ *is a subgroup of* G, *such that* $\ker f \subseteq H\}$ *and* $\mathcal{K} = \{K : K$ *is a subgroup of* $G_1\}$ *then the function* $\phi : \mathcal{H} \to \mathcal{K}$ *defined by* $\phi(H) = f(H), \forall H \in \mathcal{H}$ *is an inclusion preserving bijective map and* $\phi(H)$ *is normal in* G_1 *if and only if* H *is normal in* G.

Proof. We know that for a subgroup H of G, $f(H)$ is a subgroup of G_1.
$\implies f(H) \in \mathcal{K}$ and hence ϕ is well defined.
We now first show that ϕ is injective.
Let $\phi(H_1) = \phi(H_2)$
$\implies f(H_1) = f(H_2)$.

Let $h_1 \in H_1$

$\implies f(h_1) \in f(H_1) = f(H_2)$.

$\implies f(h_1) = f(h_2)$, for some $h_2 \in H_2$

$\implies f(h_1)f(h_2^{-1}) = e_{G_1}$

$\implies f(h_1 h_2^{-1}) = e_{G_1}$

$\implies h_1 h_2^{-1} \in \ker f \subseteq H_2$

$\implies h_1 h_2^{-1} \in H_2$

Thus $h_1 = (h_1 h_2^{-1})h_2 \in H_2$. Hence $H_1 \subseteq H_2$.

Similarly we can show that $H_2 \subseteq H_1$.

Hence $H_1 = H_2$. So ϕ is injective.

To show ϕ is surjective let us take $K \in (K)$.

Then we know that $f^{-1}(K)$ is a subgroup of G.

Now let $h \in \ker f$

$\implies f(h) = e_{G_1} \in K$ [as K is a subgroup of G_1]

$\implies h \in f^{-1}(K)$

$\implies \ker f \subseteq f^{-1}(K)$. Hence $f^{-1}(K) \in \mathcal{H}$.

Now $\phi(f^{-1}(K)) = f(f^{-1}(K)) = K$ [as f is an epimorphism] .

So ϕ is surjective and therefore ϕ is bijective.

To show ϕ is inclusion preserving map let $H_1, H_2 \in \mathcal{H}$ with $H_1 \subseteq H_2$.

Let $k \in \phi(H_1) = f(H_1)$

$\implies k = f(h_1)$ for some $h_1 \in H_1 \subseteq H_2$

$\implies k \in f(H_2) = \phi(H_2) \implies \phi(H_1) \subseteq \phi(H_2)$.

Therefore ϕ is inclusion preserving map.

Let H be a normal subgroup of G.

To show $\phi(H)$ is normal in G_1 let us take $g_1 \in G_1$ and $f(h) \in f(H) = \phi(H)$, where $h \in H$.

As f is surjective $\exists\ g \in G$ such that $f(g) = g_1$.

Now as H is normal in G

$\implies ghg^{-1} \in H$.

$\implies f(ghg^{-1}) \in f(H)$

$\implies f(g)f(h)f(g)^{-1} \in f(H)$

$\implies g_1 f(h) g_1^{-1} \in f(H)$

$\implies f(H)$ is normal in G_1, hence $\phi(H)$ is normal in G_1.

Conversely let $\phi(H)$ be a normal subgroup of G_1.

To show H is normal in G let us take two elements $g \in G$ and $h \in H$, then $f(g) \in G_1$ and $f(h) \in f(H) = \phi(H)$. As $\phi(H)$ is normal in G_1

$\implies f(g)f(h)f(g)^{-1} \in f(H)$

$\implies f(ghg^{-1}) \in f(H)$.

Thus there exists some h_0 in H such that $f(h_0) = f(ghg^{-1})$

$\implies f(ghg^{-1})f(h_0)^{-1} = e_{G_1}$

$\implies f(ghg^{-1}h_0^{-1}) = e_{G_1}$

$\implies ghg^{-1}h_0^{-1} \in \ker f \subseteq H$

$\implies ghg^{-1} = ghg^{-1}h_0^{-1}h_0 \in H$

Therefore H is normal in G.

Hence the theorem. $\square$

With the help of the above correspondence theorem one can prove the following.

Theorem 0.23. *Let K be a normal subgroup of a group G.*

(1) If H is a subgroup of G, $K \subseteq H$ then H/K is a subgroup of G/K.

(2) If T is a subgroup of G/K, then there exists a subgroup H of G such that $H/K = T$.

(3) The function ψ defined by $\psi(H) = H/K$ from the set of all subgroups of G that contains K and the set of all subgroups of G/H is a bijective function.

(4) This bijective function ψ is inclusion preserving map i.e $K \subseteq H_1 \subseteq H_2$ if and only if $H_1/K \subseteq H_2/K$

(5) H is a normal subgroup of G with $K \subseteq H$ if and only if H/K is normal subgroup of G/K.