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NAAC ACCREDITED ‘A’ GRADE

Topic: Eigen Value and Eigen Vectors

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Eigen Value of a Matrix

The roots of the characteristic equation of a square matrix A are called the eigen values (or characteristic values) of A.

- Eigen values ~~may~~ i.e. the roots of the characteristic equation may be <u>complex</u> although all the co-efficients of the characteristic equation are real

  [For real matrix i.e. matrix with real entries ~~both~~ has real co-efficients in ~~the~~ its characteristic equation]

- If λ be a multiple root of the characteristic equation of a square matrix A of multiplicity r, then λ is called an <u>r-fold</u> eigen value of A.

Illustration:- Lets find the eigen values of the matrix
$$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

$\Rightarrow$ The characteristic equation of the matrix A is

$$\cancel{|A-xI|=} \begin{vmatrix} 6-x & -2 & 2 \\ -2 & 3-x & -1 \\ 2 & -1 & 3-x \end{vmatrix} = 0$$

$$\text{or, } (6-x)\{(3-x)(3-x)-1\} + 2\{-2(3-x)+2\} + 2\{2-2(3-x)\} = 0$$

$$\text{or, } (6-x)(9-6x+x^2-1\} + 2(2x-6+2) + 2(2-6+2x) = 0$$

$$\text{or, } (6-x)(x^2-6x+8) + 4(2x-4) = 0$$

or, $(6-x)(x-4)(x-2)+8(x-2)=0$

or, $(x-2)\{(6-x)(x-4)+8\}=0$

or, $(x-2)(6x-24-x^2+4x+8)=0$

or, $(x-2)(10x-x^2-16)=0$

or, $(x-2)(x^2-10x+16)=0$

or, $(x-2)(x-8)(x-2)=0$

$\therefore$ The roots of the characteristic equation of A are 2, 2, 8.

$\therefore$ The eigen values of the matrix A are 2, 2 and 8.

Note that 2 is the multiple root of the characteristic equation of A of multiplicity 2, then 2 is an 2-fold eigen value of A.

■ Eigen Vectors of a Matrix

Definition :- Let A be a matrix (square) of order n. Any non-null vector X of n tuples is said to be eigen vector or characteristic vector of the matrix A if there exists a scalar (real or complex) λ such that $AX=\lambda X$.

[Look at that λ, it is not said that the λ must be an eigen value, it may be any scalar]

Now $AX=\lambda X \Rightarrow (A-\lambda I_n)X=0$ --- (i)

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Now (i) is a system of homogeneous equation of n ~~vari~~ variables $x_1, x_2, \cdots, x_n$ where

$X = (x_1, x_2, \cdots, x_n)^T$.

Now the system (i) has non-trivial (i.e. other than $X=0$) solution iff the co-efficient matrix of (i) is singular

i.e. $|A - \lambda I_n| = 0$

which implies that λ is nothing but the eigen values of the matrix A.

* Illustration:- We take the same matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ as in the previous illustration. The eigen values of A are 2, 2 and 8. Now we find the corresponding eigen vectors of the matrix A.

$\Rightarrow$ Eigen vectors corresponding to $\lambda = 2$

Let $X = (x_1, x_2, x_3)^T$ be an eigen vector corresponding to $\lambda = 2$. Then $AX = 2X$.

$$\text{So, } \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 6x_1 - 2x_2 + 2x_3 \\ -2x_1 + 3x_2 - x_3 \\ 2x_1 - x_2 + 3x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 6x_1 - 2x_2 + 2x_3 \\ -2x_1 + 3x_2 - x_3 \\ 2x_1 - x_2 + 3x_3 \end{pmatrix} - \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4x_1-2x_2+2x_3 \\ -2x_1+x_2-x_3 \\ 2x_1-x_2+x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So, we get three homogeneous equations

$4x_1-2x_2+2x_3=0$ i.e. $2x_1-x_2+x_3=0$

$-2x_1+x_2-x_3=0$ i.e. $2x_1-x_2+x_3=0$

$2x_1-x_2+x_3=0$ i.e. $2x_1-x_2+x_3=0$

$\therefore$ Those three equations are same i.e.

$2x_1-x_2+x_3=0$

Now if we take $x_1=K_1$ and $x_2=K_2$

then $x_3=-2K_1+K_2$

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} K_1 \\ K_2 \\ -2K_1+K_2 \end{pmatrix} = K_1\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + K_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

So, eigen vectors corresponding to $\lambda=2$ are

$K_1\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + K_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ where K_1 & K_2 are any real numbers not both 0.

(i.e. $K_1=0=K_2$ is trivial case which is excluded)

So, corresponding to eigen value 2, there are infinite numbers of eigen vectors.

Eigen vector corresponding to $\lambda=8$

Let $X=(x_1\ x_2\ x_3)^T$ be an eigen vector corresponding to $\lambda=8$. Then $AX=8X$

So, $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 8 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 6x_1 - 2x_2 + 2x_3 \\ -2x_1 + 3x_2 - x_3 \\ 2x_1 - x_2 + 3x_3 \end{pmatrix} - \begin{pmatrix} 8x_1 \\ 8x_2 \\ 8x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -2x_1 - 2x_2 + 2x_3 \\ -2x_1 - 5x_2 - x_3 \\ 2x_1 - x_2 - 5x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

which gives us three homogeneous equations as

$-2x_1 - 2x_2 + 2x_3 = 0$ i.e. $x_1 + x_2 - x_3 = 0$ ---- (i)

$-2x_1 - 5x_2 - x_3 = 0$ i.e. $2x_1 + 5x_2 + x_3 = 0$ ---- (ii)

$2x_1 - x_2 - 5x_3 = 0$ i.e. $2x_1 - x_2 - 5x_3 = 0$ ---- (iii)

(ii) − (iii) gives

$6x_2 + 6x_3 = 0 \Rightarrow x_2 + x_3 = 0$

$\Rightarrow x_2 = -x_3$

Putting $x_2 = -x_3$ in (i) we get

$x_1 - x_3 - x_3 = 0$ i.e. $x_1 = 2x_3$

Put $x_1 = 2x_3$ in (ii), we get $4x_3 + 5x_2 + x_3 = 0$

$\Rightarrow x_2 = -x_3$

Putting $x_1 = 2x_3$ in (iii) we again get $x_2 = -x_3$

Let us take $x_2 = K$, then $x_3 = -K$, then $x_1 = -2K$

so, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2K \\ K \\ -K \end{pmatrix} = K \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

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So, the eigen vectors of A, corresponding to $\lambda=8$ are $K\begin{pmatrix}-2\\1\\-1\end{pmatrix}$ where K is any real number except 0 [i.e. $K\neq 0$]

* [Note:- Your book has got different answer, they got the eigen vectors as $K\begin{pmatrix}2\\-1\\1\end{pmatrix}$.

Don't worry, they are same as K is any real number, if you choose negative value of K in the book you will get the eigen vectors corresponding to the positive values of K in my answer.

Again if you choose positive values of K in the book you will get the eigen vectors corresponding to the negative values of K in my answer.

As an example if you choose $K=2$ in book you will get the eigen vector as $2\begin{pmatrix}2\\-1\\1\end{pmatrix}$ which is same eigen ~~value~~ vector if you choose $K=-2$ in my answer.

So, the answer is same] *

$\therefore$ Again you get an infinite number of eigen vectors corresponding to $\lambda=8$.

* Find the eigen values and eigen vectors of the 3X3 matrix $A = \begin{pmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{pmatrix}$

Ans:- The characteristic equation of the matrix A is

$$|A - xI| = 0$$

$$\text{or, } \begin{vmatrix} 8-x & -8 & -2 \\ 4 & -3-x & -2 \\ 3 & -4 & 1-x \end{vmatrix} = 0$$

$$\text{or, } (8-x)\{(-3-x)(1-x) - 8\} + 8\{4(1-x)+6\} - 2\{-16-(-3-x)3\} = 0$$

$$\text{or, } (8-x)\{(-3+3x-x+x^2)-8\} + 8\{4-4x+6) - 2(-16+9+3x) = 0$$

$$\text{or, } (8-x)(x^2+2x-11) + 8(-4x+10) - 2(3x-7) = 0$$

$$\text{or, } 8x^2+16x-88-x^3-2x^2+11x-32x+80-6x+14 = 0$$

$$\text{or, } -x^3+6x^2-11x+6 = 0$$

$$\text{or, } x^3-6x^2+11x-6 = 0$$

$$\text{or, } x^3-x^2-5x^2+5x+6x-6 = 0$$

$$\text{or, } x^2(x-1)-5x(x-1)+6.(x-1) = 0$$

$$\text{or, } (x-1)(x^2-5x+6) = 0$$

$$\text{or, } (x-1)(x-2)(x-3) = 0$$

$\therefore x = 1, 2, 3$ (roots of the characteristic equation)

$\therefore$ The eigen values of the matrix A are 1, 2 and 3.

i.e. $\lambda = 1, 2, 3$ are the eigen values of A.

Now we will calculate the eigen vectors.

Eigen vector corresponding to $\lambda = 1$

Let $X = (x_1 \ x_2 \ x_3)^T$ be the eigen vector corresponding to the eigen value $\lambda = 1$. Then

$$AX = 1 \cdot X$$

or, $$\begin{pmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1 \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or, $$\begin{pmatrix} 8x_1 - 8x_2 - 2x_3 \\ 4x_1 - 3x_2 - 2x_3 \\ 3x_1 - 4x_2 + x_3 \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or, $$\begin{pmatrix} 7x_1 - 8x_2 - 2x_3 \\ 4x_1 - 4x_2 - 2x_3 \\ 3x_1 - 4x_2 + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Which gives us the homogeneous equations

$$7x_1 - 8x_2 - 2x_3 = 0 \quad \text{---- (1.a)}$$

$$4x_1 - 4x_2 - 2x_3 = 0 \quad \text{---- (1.b)}$$

$$3x_1 - 4x_2 = 0 \quad \text{---- (1.c)}$$

From equation (1.c) we get $3x_1 = 4x_2$

Putting this value in the equation (1.b) we get

$$4x_1 - 3x_1 - 2x_3 = 0$$

or, $x_1 = 2x_3$

Now (1.b) + (1.c) = (1.a), So, we can not get any particular solution from those three equations.

We have $x_1 = 2x_3$ & $3x_1 = 4x_2$

If we take $x_3 = K$, then $x_1 = 2K$

Now if $x_1 = 2K$, then $3(2K) = 4x_2$

or, $6K = 4x_2$

or, $x_2 = \frac{6}{4}K$

Eigen vectors are

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2K \\ 6/4\,K \\ K \end{pmatrix} = K\begin{pmatrix} 2 \\ 6/4 \\ 1 \end{pmatrix} = K\begin{pmatrix} 2 \\ 3/2 \\ 1 \end{pmatrix} = \frac{K}{2}\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

where K is any real number other than 0.

[i.e. $K \neq 0$]

Eigen vector corresponding to the eigen value $\lambda = 2$

Let $X = (x_1 \; x_2 \; x_3)^T$ is the eigen vector corresponding to the eigen value $\lambda = 2$

Then $AX = 2X$

$$\text{or, } \begin{pmatrix} 8x_1 - 8x_2 - 2x_3 \\ 4x_1 - 3x_2 - 2x_3 \\ 3x_1 - 4x_2 + x_3 \end{pmatrix} - 2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{or, } \begin{pmatrix} 6x_1 - 8x_2 - 2x_3 \\ 4x_1 - 5x_2 - 2x_3 \\ 3x_1 - 4x_2 - x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which gives us the equations

$6x_1 - 8x_2 - 2x_3 = 0$ ---- (2.a)

$4x_1 - 5x_2 - 2x_3 = 0$ ---- (2.b)

$3x_1 - 4x_2 - x_3 = 0$ ---- (2.c)

Now. $2\times(2.c) = (2.a)$ $\therefore$ the equations $(2.a)$ & $(2.c)$ are same.

From equation $(2.c)$ we get $x_3 = 3x_1 - 4x_2$

Putting this value in equation $(2.b)$ we get

$$4x_1 - 5x_2 - 2(3x_1 - 4x_2) = 0$$

or, $4x_1 - 5x_2 - 6x_1 + 8x_2 = 0$

or, $-2x_1 + 3x_2 = 0$

or, $2x_1 = 3x_2$ or, $x_1 = \frac{3}{2}x_2$

$\therefore$ $x_3 = 3\left(\frac{3}{2}\right)x_2 - 4x_2$

or, $x_3 = \frac{9}{2}x_2 - 4x_2$

or, $x_3 = \frac{1}{2}x_2$

So, if we put $x_3 = K$ then $x_2 = 2K$

So, $x_1 = \frac{3}{2}(2K) = 3K$

$\therefore$ The eigen vectors corresponding to $\lambda = 2$ are

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3K \\ 2K \\ K \end{pmatrix} = K\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

where K is any real number except 0, [i.e. $K \neq 0$]

To Calculate the eigen vector corresponding to $\lambda = 3$ we will get

$$\begin{pmatrix} 8x_1 - 8x_2 - 2x_3 \\ 4x_1 - 3x_2 - 2x_3 \\ 3x_1 - 4x_2 + x_3 \end{pmatrix} - 3\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or, $\begin{pmatrix} 5x_1 - 8x_2 - 2x_3 \\ 4x_1 - 6x_2 - 2x_3 \\ 3x_1 - 4x_2 - 2x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

which gives the equations

$$5x_1 - 8x_2 - 2x_3 = 0 \quad \cdots\cdots (3.a)$$
$$4x_1 - 6x_2 - 2x_3 = 0 \quad \cdots\cdots (3.b)$$
$$3x_1 - 4x_2 - 2x_3 = 0 \quad \cdots\cdots (3.c)$$

$(3.b) - (3.c)$ gives

$$x_1 - 2x_2 = 0$$
$$\text{or, } x_1 = 2x_2$$

Putting this value to $(3.c)$ we get

$$6x_2 - 4x_2 - 2x_3 = 0$$
$$\text{or, } 2x_2 = 2x_3$$
$$\text{or, } x_2 = x_3$$

$2 \times (3.b) - (3.c) = (3.a)$

$\therefore$ Those equations will not give us any particular solutions.

$\therefore$ Let $x_3 = K$, then $x_2 = K$

$\therefore x_1 = 2 \cdot K = 2K$

$\therefore$ The eigen vectors corresponding to $\lambda = 3$ are

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2K \\ K \\ K \end{pmatrix} = K \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

where K is any real number except 0 [i.e. $K \neq 0$]

Some times you get to see that the eigen vector is $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, where K is not mentioned.

But if X is an eigen ~~value~~ vector then any non-zero constant multiplied ~~by~~ with X is also an eigen vector.

So, you can write that $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is the eigen vector corresponding to the eigenvalue $\lambda = 3$, which is not wrong, but writting the eigen vector as $K\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ $[K \in \mathbb{R}, K \neq 0]$ is more general.