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GROUP HOMOMORPHISM

(Isomorphism Theorems)

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Theorem 0.17. *Natural Homomorphism:*

Let H be a normal subgroup of a group G then the mapping $\theta : G \to G/H$ defined by $\theta(x) = xH$, $x \in G$ is an onto homomorphism with kernel H.

Proof. To show θ is a homomorphism let us take two elements x, y from G. Then $\theta(x) = xH$ and $\theta(y) = yH$.

Now $\theta(xyH) = xyH = (xH)(yH) = \theta(x)\theta(y)$. This shows that θ is a homomorphism.

The identity element in the quotient group G/H is H,.

If $x \in \ker\theta \Leftrightarrow \theta(x) = H \Leftrightarrow xH = H \Leftrightarrow x \in H$. Therefore $\ker\theta = H$.

Again for any $aH \in G/H$ has a preimage a in G as $\theta(a) = aH$, thus θ is surjective. Hence the result. $\square$

Remark 0.18. This homomorphism θ is said to be the natural homomorphism from G onto G/H.

Thus in conclution we can say that for any normal subgroup H in G there always exists a homomorphism θ, the natural homomorphism from G onto G/H. In particular when $H = G$ then θ becomes the trivial homomorphism.

Theorem 0.19. *First Isomorphism Theorem:*

Let G and G' be two groups and $\phi : G \to G'$ be an onto homomorphism. Then the quotient group $G/\ker\phi \simeq G'$

Proof. Let $H = \ker\phi$, then H is a normal subgroup of G, so G/H is defined.

Let us define a mapping $\psi : G/H \to G'$ by $\psi(aH) = \phi(a)$, $aH \in G/H$.

We now first show that ψ is well defined.

$aH = a'H$

$\Rightarrow a^{-1}a' \in H$

$\Rightarrow \phi(a^{-1}a') = e_{G'}$, [as $H = \ker\phi$]

$\Rightarrow \{\phi(a)\}^{-1}\phi(a') = e_{G'}$

$\Rightarrow \phi(a) = \phi(a')$

$\Rightarrow \psi(aH) = \psi(a'H)$.

Therefore ψ is well defined.

To show ψ is a homomorphism let us take two elements aH and bH.

$\psi(aHbH) = \psi(abH) = \phi(ab) = \phi(a)\phi(b) = \psi(aH)\psi(bH)$. Thus ψ is a homomorphism.

Now ψ is one-one because

$\psi(aH) = \psi(bH)$

$\Rightarrow \phi(a) = \phi(b)$

$\Rightarrow \phi(a)^{-1}\phi(b) = e_{G'}$

$\Rightarrow \phi(a^{-1}b) = e_{G'}$

$\Rightarrow a^{-1}b \in \ker\phi = H$

$\Rightarrow aH = bH$.

Again ϕ is onto, because each element of G' is of the form $\phi(a)$ for some $a \in G$ and as

$\phi(a) = \psi(aH)$, the preimage of $\phi(a)$ is aH in G/H.

Thus ψ is an isomorphism and hence $G/H \simeq G'$ i.e $G/\ker\phi \simeq G'$. $\square$

Theorem 0.20. *Second Isomorphism theorem:*
Let H and K are two subgroups of a group G with K normal in G.
Then $H/(H \cap K) \simeq HK/K$.

Proof. Since K is a normal subgroup of G, $HK = \{hk : h \in H, k \in K\}$ is a subgroup of G with $K \subseteq HK \subseteq G$.

Again K is normal in $G \implies K$ is normal in HK. Hence HK/K is exists.

Now define a mapping $f : H \to HK/K$ by $f(h) = hK, \forall h \in H$.

Let $f(h_1h_2) = h_1h_2K = (h_1K)(h_2K) = f(h_1)f(h_2)$. Thus f is a homomorphism.

Let $aK \in HK/K$,then $\exists$ $h_0 \in H$ and $k_0 \in K$ such that $a = h_0k_0$.

Now $aK = h_0k_0K = h_0K = f(h_0)$.[By definition of f]. Hence f is an epimorohism.

Hence by First Isomorphism theorem $H/\ker f \simeq HK/K$

Now $\ker f = \{h \in H : f(h) = K\}$

$= \{h \in H : hK = K\}$

$= \{h \in H : h \in K\}$

$= H \cap K.$

Thus $H/(H \cap K) \simeq HK/K.$ $\square$

Theorem 0.21. *Third Isomorphism Theorem:*
Let H_1 and H_2 be two normal subgroups of a group G such that $H_1 \subseteq H_2$. Then $(G/H_1)/(H_2/H_1) \simeq (G/H_2)$.

Proof. To prove the theorem let us define a mapping
$f : G/H_1 \to G/H_2$ by
$f(aH_1) = aH_2, \forall a \in G.$
We now first show that f is well defined.
$aH_1 = bH_1$
$\implies a^{-1}b \in H_1 \subseteq H_2$
$\implies aH_2 = bH_2$
$\implies f(aH_1) = f(bH_1).$
Thus f is well-defined.

Next to show f is a homomorphism let us take two elements aH_1 and bH_1 from G/H_1.

Then $f(aH_1bH_1) = f(abH_1) = abH_2 = (aH_2)(bH_2) = f(aH_1)f(bH_1)$.

Hence f is a homomorphism.

Again let $cH_2 \in G/H_2$ and then $f(cH_1) = cH_2$. Thus f is an epimorphism.

By first Isomorphism theorem $(G/H_1)/\ker f \simeq G/H_2$.

Now $\ker f = \{aH_1 \in G/H_1 : f(aH) = H_2\}$

$= \{aH_1 \in G/H_1 : aH_2 = H_2\}$

$= \{aH_1 \in G/H : a \in H_2\}$

$= H_2/H_1$.

Therefore $(G/H_1)/(H_2/H_1) \simeq G/H_2$. $\square$