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POWER SERIES

(Abel's Theorem)

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Theorem 0.20. *Abel's Theorem :*

Let $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with radius of convergence* $R > 0$. *If the series converges at the end point* R *of the interval of convergence* $(-R, R)$, *then the series is uniformly convergent on the closed interval* $[0, R]$

Again if the series converges at the end point $-R$ *of the interval of convergence* $(-R, R)$, *then the series is uniformly convergent on the closed interval* $[-R, 0]$.

Proof. Let $\sum_{n=0}^{\infty} a_n x^n$ converges at R i.e $\sum_{n=0}^{\infty} a_n R^n$ is convergent. Let us take $\epsilon > 0$. Then $\exists$ $k \in \mathbb{N}$ such that $|a_{n+1}R^{n+1} + a_{n+2}R^{n+2} + \cdots + a_{n+p}R^{n+p}| < \epsilon$, $\forall n \geq k$, $p = 1, 2, 3, \ldots$

Let $s_{n,r} = |a_{n+1}R^{n+1} + a_{n+2}R^{n+2} + \cdots + a_{n+r}R^{n+r}|$, $r = 1, 2, 3, \ldots$.

Then $|s_{n,p}| < \epsilon$, $\forall n \geq k$, $p = 1, 2, \ldots$

Let $x \in [0, R]$. Then $|a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots + a_{n+p}x^{n+p}|$
$= |a_{n+1}R^{n+1}(\frac{x}{R})^{n+1} + a_{n+2}R^{n+2}(\frac{x}{R})^{n+2} + \cdots + a_{n+p}R^{n+p}(\frac{x}{R})^{n+p}|$
$= |s_{n,1}(\frac{x}{R})^{n+1} + (s_{n,2} - s_{n,1})(\frac{x}{R})^{n+2} + \cdots + (s_{n,p} - s_{n,p-1})(\frac{x}{R})^{n+p}|$
$= |s_{n,1}\{(\frac{x}{R})^{n+1} - (\frac{x}{R})^{n+2}\} + s_{n,2}\{(\frac{x}{R})^{n+2} - (\frac{x}{R})^{n+3}\} + \cdots + s_{n,p-1}\{(\frac{x}{R})^{n+p-1} - (\frac{x}{R})^{n+p}\} + s_{n,p}(\frac{x}{R})^{n+p}|$
$\leq |s_{n,1}||(\frac{x}{R})^{n+1} - (\frac{x}{R})^{n+2}| + |s_{n,2}||(\frac{x}{R})^{n+2} - (\frac{x}{R})^{n+3}| + \cdots + |s_{n,p-1}||(\frac{x}{R})^{n+p-1} - (\frac{x}{R})^{n+p}| + |s_{n,p}||(\frac{x}{R})^{n+p}|$
$\leq \epsilon\{(\frac{x}{R})^{n+1} - (\frac{x}{R})^{n+2} + (\frac{x}{R})^{n+2} - (\frac{x}{R})^{n+3} + \cdots + (\frac{x}{R})^{n+p-1} - (\frac{x}{R})^{n+p} + (\frac{x}{R})^{n+p}\}$, $\forall n \geq k$, $p = 1, 2, \ldots$
$= \epsilon(\frac{x}{R})^{n+1} \leq \epsilon, \forall n \geq k,\ p = 1, 2, \ldots$
Thus $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, R]$ by Cauchy criterion.
Similarly we can proof the second part of the theorem. □

Remark 0.21. If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R and the series is convergent at both the end points, then the power series is uniformly convergent on $[-R, R]$.

Theorem 0.22. *Abel's Theorem (Another form)*

Let $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with radius of convergent* 1.

If $\sum_{n=0}^{\infty} a_n$ *be convergent the the series* $\sum_{n=0}^{\infty} a_n x^n$ *is uniformly convergent on* $[0, 1]$

Again if $\sum_{n=0}^{\infty} (-1)^n a_n$ *be convergent then the series* $\sum_{n=0}^{\infty} a_n x^n$ *is uniformly convergent on* $[-1, 0]$.

Proof. Similar proof. □

Theorem 0.23. *Abel's Theorem (Limit form)*

Let $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with radius of convergent* R *and let the sum of the series be* $f(x)$ *on* $(-R, R)$*. If the series* $\sum_{n=0}^{\infty} a_n R^n$ *be convergent then* $\sum_{n=0}^{\infty} a_n R^n = \lim_{x \to R^-} f(x)$*.*

Proof. Since R is the radius of convergent of the series and $\sum_{n=0}^{\infty} a_n R^n$ is convergent, the series is convergent uniformly on $[0, R]$. Let $\phi(x)$ be the sum of the series on $[0, R]$. As each term of the series is continuous on $[0, R]$, the sum function ϕ is also continuous on $[0, R]$. Also $\phi(x) = f(x)$ on $[0, R)$. Since ϕ is continuous at R, $\phi(R) = \lim_{x \to R^-} \phi(x) = \lim_{x \to R^-} f(x)$.

Therefore $\sum_{n=0}^{\infty} a_n R^n = \lim_{x \to R^-} f(x)$. □

Corollary 0.24. *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with radius of convergent* 1 *and let the sum of the series be* $f(x)$ *on* $(-1,1)$. *Then*

(1) if the power series be convergent at 1, *then* $\sum_{n=0}^{\infty} a_n = \lim_{x\to 1^-} f(x)$

(2) if the power series is convergent at -1, *then* $\sum_{n=0}^{\infty} (-1)^n a_n = \lim_{x\to -1^+} f(x)$

Note 0.25. Converse of the Abel's theorem is not true. For a power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence R, $\lim_{x\to R-} f(x)$ may exists but the series $\sum_{n=0}^{\infty} a_n x^n$ may not converge at R.

For example let us take the series $1-x+x^2-x^3+\cdots$ whosh radius of convergent is 1. Then sum of the series is $\frac{1}{x+1}$ on $(-1,1)$ and $\lim_{x\to 1^-} \frac{1}{x+1} = 2$ but the series $1-x+x^2-x^3+\cdots$ is not convergent at $x=1$.

Example 0.26. Find the sum of the power series $1 + x + x^2 + \cdots$ on its inerval of convergence. Deduce the power series expansion of $\log(1 - x)$ and use Abel's theorem to prove that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$.

Solution: Let $\sum_{n=0}^{\infty} a_n x^n$ be the power series. Then $a_n = 1$, $n = 0, 1, 2, \ldots$.
Therefore $\overline{\lim}|a_n|^{\frac{1}{n}} = 1$, so the radius of convergence of the power series is 1 and hence the series is converges absoluely in $(-1, 1)$.
Let $s_n(x) = 1 + x + \cdots + x^{n-1}$, $|x| < 1$.
Then $s_n(x) = \frac{1-x^n}{1-x}$, $|x| < 1 \implies \lim_{n\to\infty} s_n(x) = \dfrac{1}{1-x}$, $|x| < 1$.
We know that a power series can be integrated term by term in any closed interval contained in $(-1, 1)$. Let us take x with $|x| < 1$. Integrating the power series term by term on $[0, x]$ or $[x, 0]$ we get

$\int_0^x \frac{dx}{1-x} = \int_0^x dx + \int_0^x x dx + \int_0^x x^2 dx + \cdots$, with $|x| < 1$.

or $-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ with $|x| < 1$

or $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots$ with $|x| < 1$........(1)

The power series on the right hand side at $x = 1$ is $-(1 + \frac{1}{2} + \frac{1}{3} + \cdots)$ is divergent and for $x = -1$, the power series becomes $1 - \frac{1}{2} + \frac{1}{3} - \cdots$ is convergent.

$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots, -1 \leq x < 1$. Since the power series in (1) converges to the sum $\log(1-x)$ in $(-1, 1)$ and the power series converges at $x = -1$, by Abel's test $1 - \frac{1}{2} + \frac{1}{3} - \cdots = \lim_{x \to -1^+} \log(1-x) = \log 2$.