VIVEKANANDA COLLEGE
THAKURPUKUR
KOLKATA-700063

NAAC ACCREDITED ‘A’ GRADE

Topic: Power Series

Course Title: Radius of Convergence
And Cauchy Hadamard Theorem

Paper: CC8

Unit: 3

Semester: 4

Name of the Teacher: Jayeeta Saha

Name of the Department: Mathematics

MATHEMATICS

(HONOURS)

SEM-IV(CC8,UNIT-3)

POWER SERIES

(Radius of Convergence

&

Cauchy Hadamard Theorem)

Dr. JAYEETA SAHA

ASSISTANT PROFESSOR
DEPARTMENT OF MATHEMATICS,
VIVEKANANDA COLLEGE, THAKURPUKUR

Power Series

A series of the form $a_0 + a_1x + a_2x^2 \cdots$ where $a_0, a_1, a_2, \cdots$ are real numbers is called power series of real numbers.

The general form of the power series is $a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$ where $a_0, a_1, a_2, \cdots \in \mathbb{R}$,which is called a power series about the point x_0. This general form can be transferred to the form $a_0 + a_1x + a_2x^2 \cdots$ through the substitution $y = x - x_0$. Therefore to study the nature and properties of a power series one can consider only the form $a_0 + a_1x + a_2x^2 \cdots$, which is denoted by $\sum_{n=0}^{\infty} a_n x^n$.

Example of power series:

1. $1 + x + x^2 + x^3 + \cdots$
2. $\frac{1}{2} + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \cdots$
3. $\sum_{n=0}^{\infty} n!x^n$

The first thing to notice about a power series is that it is a function of x. That is different from any other kind of series that we have looked at to this point.

Some power series are convergent everywhere whereas some power series are convergent in an interval. There also exist nowhere convergent power series which are convergent only for $x = 0$.

Theorem 0.1. *If a power series* $\sum_{n=0}^{\infty} a_n x^n$ *converges for* $x = x_1$, *then the series converges absolutely for all real* x *with* $|x| < |x_1|$

Proof. The series is convergent for $x = x_1 \implies \sum_{n=0}^{\infty} a_n x_1^n$ is convergent.

This shows that $\lim a_n x_1^n = 0 \implies$ the sequence $\{a_n x_1^n\}$ is bounded.

So there exists a real number k such that $|a_n x_1^n| \leq k$, $\forall n \in \mathbb{N}$.

Now $|a_n x^n| = |a_n x_1^n|.|\frac{x}{x_1}|^n \leq k|\frac{x}{x_1}|^n$.

By comparison test $\sum_{n=0}^{\infty} |a_n x^n|$ is convergent series for all x satisfying $|\frac{x}{x_1}| < 1$ i.e $|x| < |x_1|$.

Hence the theorem. □

Note 0.2. It can be easily proof that if a power series diverges for $x = x_1$, then the series diverges for all real x with $|x| > |x_1|$.

Theorem 0.3. *If a power series* $\sum_{n=0}^{\infty} a_n x^n$ *is neither no where convergent nor everywhere convergent then* $\exists$ *a positive real number* R *such that the series is absolutely convergent for all* x *satisfying* $|x| < R$ *and divergent for all* x *satisfying* $|x| > R$

The power series is neither no where convergent nor every where convergent.

$\exists\, c\,(\neq 0), d \in \mathbb{R}$ such that the series converges at c and diverges at d.

Let $0 < c_1 < |c|$ and $d_1 > |d| > 0$. Then the power series converges at c_1 and diverges at d_1, clearly $0 < c_1 < d_1$.

Let $I_1 = [c_1, d_1]$. Let $c_1' = \frac{1}{2}(c_1 + d_1)$.

If c_1' be a point of convergence of the power series we take $[c_1', d_1]$, otherwise we take $[c_1, c_1']$. Let this interval be $I_2 = [c_2, d_2]$. $|I_2| = \frac{1}{2}(d_1 - c_1)$

Let $c_2' = \frac{1}{2}(c_2 + d_2)$. If c_2' is a point of convergence of the power series we take $[c_2', d_2]$, otherwise we take $[c_2, c_2']$.

Let this interval be $I_3 = [c_3, d_3]$, $|I_3| = \frac{1}{2^2}(d_1 - c_1)$

Continuing this process we obtain a sequence $\{I_n\}_n$ with the properties :

(i) $I_n = [c_n, d_n]$, $|I_n| = \frac{1}{2^{n-1}}(d_1 - c_1) \to 0$ as $n \to \infty$

(ii) $I_{n+1} \subset I_n \;\forall\, n$ and

(iii) c_n is a point of convergence and d_n is a point of divergence of the power series.

Then by the Cantor's intersection theorem, ∃ a unique ρ such that $\rho \in \bigcap_{n=1}^{\infty} I_n$.

$\therefore\ c_n \le \rho \le d_n\ \forall\, n$ and $\sup_n c_n = \rho = \inf_n d_n$

Let x' be such that $0 < x' < \rho$. Then $\exists\, m \in \mathbb{N}$ such that $x' < c_m \le \rho$

⇒ the power series converges at x'

⇒ the power series converges absolutely $\forall\, x$ satisfying $|x| < x'$.

Since x' is arbitrary, the power series converges absolutely $\forall\, x$ satisfying $|x| < \rho$

Let x'' be such that $x'' > \rho$. Then $\exists\, k \in \mathbb{N}$ s.t. $\rho \le d_k < x''$

Since the power series diverges at d_k and $x'' > d_k$, the power series diverges at x''

⇒ the power series diverges $\forall\, x$ satisfying $|x| > x''$

Since $x'' > \rho$ is arbitrary, the power series diverges $\forall\, x$ satisfying $|x| > \rho$.

Hence $\rho = R$ and the theorem is proved.

Definition : (Radius of convergence) : Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series which is neither no where convergent nor every where convergent, then $\exists$ a +ve real number ρ such that the series is absolutely convergent $\forall$ x satisfying $|x| < \rho$ and the series is divergent $\forall$ x satisfying $|x| > \rho$. Then ρ is called the radius of convergence of the power series. The interval $(-\rho, \rho)$ is called interval of convergence of the power series.

Theorem 0.4. *Cauchy -Hadamard Theorem:*

Let $\sum_{n=0}^{\infty} a_n x^n$ *be a power series and* $\overline{\lim}|a_n|^{\frac{1}{n}} = \mu$*.Thenn*

(i) $0 < \mu < \infty \implies$ *the powerr series is absolutely convergent for all* x *satisfying* $|x| < \frac{1}{\mu}$ *and is divergentfor all* x *satisfying* $|x| > \frac{1}{\mu}$

(ii) $\mu = 0 \implies$ *the series is everywhere convergent.*

(iii) $\mu = +\infty \implies$ *the series is no where convergent i,e everywhere divergent.*

Proof : (i) Let $0 < \mu < \infty$

Let $u_n = a_n x^n$, $n = 0, 1, 2, \ldots$

Then $\lim_{n\to\infty} |u_n|^{1/n} = \lim_{n\to\infty} |a_n|^{1/n} |x| = \mu |x|$

$\therefore \sum_{n=0}^{\infty} u_n$ converges absolutely if $\mu |x| < 1$, i.e. if $|x| < \frac{1}{\mu}$ [Cauchy's root test]

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ converges absolutely $\forall\, x$ satisfying $|x| < \frac{1}{\mu}$

If $\mu |x| > 1$, then $\overline{\lim} |a_n x^n|^{1/n} = \overline{\lim} |a_n|^{1/n} |x| = \mu |x| > 1$

Then $\overline{\lim} |u_n|^{1/n} > 1 \Rightarrow \overline{\lim} |u_n| > 1$

$\therefore \lim |u_n| \neq 0 \Rightarrow \lim u_n \neq 0$

$\therefore \sum_{n=0}^{\infty} u_n$ is divergent

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ is divergent if $|x| \mu > 1$, i.e. $|x| > \frac{1}{\mu}$

(ii) $\mu = 0$

Let $\alpha \neq 0$ and $\varepsilon = \frac{1}{2|\alpha|}$

Now $\overline{\lim} |a_n|^{1/n} = 0$

$\Rightarrow \exists\, k \in \mathbb{N}$ such that $|a_n|^{1/n} < \varepsilon \;\forall\; n \geq k$

$\Rightarrow |a_n\, \alpha^n| < \dfrac{1}{2^n} \;\forall\; n \geq k$

$\Rightarrow \displaystyle\sum_{n=0}^{\infty} |a_n \alpha^n|$ is convergent by comparison test, since $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$ is convergent.

$\Rightarrow \displaystyle\sum_{n=0}^{\infty} a_n \alpha^n$ is absolutely convergent.

Since $\alpha\ (\neq 0)$ is arbitrary, $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ is divergent everywhere.

(iii) $\mu = +\infty$

If possible, let $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ be convergent at some $\beta\ (\neq 0)$.

Then $\displaystyle\sum_{n=0}^{\infty} a_n \beta^n$ is convergent.

$\Rightarrow \{a_n\, \beta^n\}_n$ is bounded sequence of real numbers.

$\Rightarrow \exists\, k > 0$ such that $|a_n \beta^n| < k \;\forall\, n = 0, 1, 2, \ldots$

$\Rightarrow |a_n|^{1/n} < \dfrac{k^{1/n}}{|\beta|}$

$\Rightarrow \{|a_n|^{1/n}\}_n$ is bounded, since $\left\{\dfrac{k^{1/n}}{|\beta|}\right\}_n$ is convergent and hence bounded.

$\Rightarrow \overline{\lim}\, |a_n|^{1/n}$ is finite, a contradiction.

$\therefore \sum_{n=0}^{\infty} a_n \beta^n$ is convergent at some $\beta\,(\neq 0)$ is not true.

$\therefore$ the series $\sum_{n=0}^{\infty} a_n x^n$ is everywhere divergent.

Note: The radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is given by $r = \frac{1}{\overline{\lim}|a_n|^{\frac{1}{n}}} = \frac{1}{\mu}$

Theorem 0.5. *Ratio Test:*

Let $\sum_{n=0}^{\infty} a_n x^n$ *be a power series and* $\lim |\frac{a_{n+1}}{a_n}| = \mu$, *then*

(i) $0 < \mu < \infty \implies$ *the power series is absolutely convergent for all* x *satisfying* $|x| < \frac{1}{\mu}$ *and is divergentfor all* x *satisfying* $|x| > \frac{1}{\mu}$

(ii) $\mu = 0 \implies$ *the series is everywhere convergent.*

(iii) $\mu = +\infty \implies$ *the series is no where convergent i,e everywhere divergent.*

Proof : (i) Let $0 < \mu < \infty$

Let $u_n = a_n x^n$, $n = 0, 1, 2, \ldots$

Then, $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x| = \mu|x|$

$\therefore$ by D' Alembert's ratio test, $\sum_{n=0}^{\infty} |u_n|$ is convergent if $\mu|x| < 1$ i.e. if $|x| < \frac{1}{\mu}$

$\therefore$ $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely $\forall$ x satisfying $|x| < \frac{1}{\mu}$

If $\mu|x| > 1$, then $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| > 1$

Let $\varepsilon > 0$ be such that $l - \varepsilon > 1$, where $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = l$

Then $\exists\, m \in \mathbb{N}$, $l - \varepsilon < \left|\frac{u_{n+1}}{u_n}\right| < l + \varepsilon \;\forall\; n \geq m$

$\Rightarrow \left|\frac{u_{n+1}}{u_n}\right| > l - \varepsilon > 1 \;\forall\; n \geq m$

$\Rightarrow |u_{n+1}| > |u_n| \;\forall\; n \geq m$

$\Rightarrow \{|u_n|\}_n$ is ultimately monotonic increasing.

$\Rightarrow \lim_{n\to\infty} |u_n| \neq 0$ and so $\lim_{n\to\infty} u_n \neq 0$

$\therefore \sum_{n=0}^{\infty} u_n$ is divergent.

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ is divergent $\forall\, x$ satisfying $|x| > \frac{1}{\mu}$

(ii) $\mu = 0$

Let $x \neq 0$ and $u_n = a_n x^n$, $n = 0, 1, 2, \ldots$

Then $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x| = 0 < 1$

$\therefore$ by D' Alembert's ratio test, $\sum_{n=0}^{\infty} |u_n|$ is convergent.

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent $\forall\, x(\neq 0) \in \mathbb{R}$

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ is convergent everywehre in $\mathbb{R}$.

(iii) $\mu = +\infty$

If possible, let $\sum_{n=0}^{\infty} a_n x^n$ be convergent at some α ($\neq 0$)

Then $\sum_{n=0}^{\infty} a_n \alpha^n$ is convergent.

Let $u_n = a_n \alpha^n$, $n = 0, 1, 2, \ldots$

Then $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |\alpha| = \infty$

Let $G > 1$ be arbitrary. Then $\exists\, m \in \mathbb{N}$ such that $\left|\frac{u_{n+1}}{u_n}\right| > G > 1 \;\forall\, n \geq m$

$\Rightarrow |u_{n+1}| > |u_n| \;\forall\, n \geq m$

$\Rightarrow \{|u_n|\}_n$ is ultimately monotone increasing sequence of +ve real numbers.

$\Rightarrow \lim_{n\to\infty} |u_n| \neq 0$ and so $\lim_{n\to\infty} u_n \neq 0$

$\therefore \sum_{n=0}^{\infty} u_n$, i.e. $\sum_{n=0}^{\infty} a_n \alpha^n$ is not convergent, a contradiction to our supposition.

$\therefore \sum_{n=0}^{\infty} a_n x^n$ converges at some non zero x is not true.

$\therefore \sum_{n=0}^{\infty} a_n x^n$ is divergent everywhere.

Note: The radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is given by $r = \frac{1}{\lim|\frac{a_{n+1}}{a_n}|} = \frac{1}{\mu}$

Remark:

$$\underline{\lim} \left|\frac{a_{n+1}}{a_n}\right| \le \underline{\lim} \ |a_n|^{1/n} \le \overline{\lim} \ |a_n|^{1/n} \le \overline{\lim} \left|\frac{a_{n+1}}{a_n}\right|$$

if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists then $\lim_{n\to\infty} |a_n|^{1/n}$ exists, but the converse is not true.

From this it follows that if ratio test is applicable to a series then Cauchy-Hadamard test is applicable while there are series where Cauchy-Hadamard is applicable but the ratio test fail to be applicable.

Let $a_n = \frac{3+(-1)^n}{2}$, $n = 1, 2, \ldots$

The sequence is $\{1, 2, 1, 2, 1, 2, \ldots\}$

Here $\lim a_n^{1/n} = 1$, since $\lim a_{2n}^{1/2n} = \lim 2^{1/2n} = 1$ and $\lim a_{2n-1}^{1/2n-1} = \lim 1^{\frac{1}{2n-1}} = 1$

But $\lim \frac{a_{n+1}}{a_n}$ does not exist.