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GROUP HOMOMORPHISM

(SOME THEOREMS AND RESULTS ON GROUP HOMOMORPHISM)

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Theorem 0.1. *Let $(G, \cdot)$ and $(G', *)$ be two groups and $\phi : G \to G'$ be a homomorphism. Then the followings are hold.*

(1) If H be a subgroup of G, $\phi(H)$ is a subgroup of G'.

(2) If K be a subgroup of G', $\phi^{-1}(K) = \{x \in G : \phi(x) \in K\}$ is a subgroup of G.

Proof. (1)As H be a subgroup of G , $e_G \in H$, hence $\phi(e_G) \in \phi(H)$. This shows that $\phi(H)$ is non-empty.

Let us take two elements $a', b' \in \phi(H)$. Then there exists two elements say $a, b \in H$ such that $\phi(a) = a'$ and $\phi(b) = b'$. Now $a' * (b')^{-1} = \phi(a) * \phi(b)^{-1} = \phi(a \cdot b^{-1}) \in \phi(H)$, as H is a subgroup. Hence $\phi(H)$ is a subgroup of G'.

(2) Similar proof. □

Theorem 0.2. *Let $(G, \cdot)$ and $(G', *)$ be two groups and $\phi : G \to G'$ be an onto homomorphism. Then*
(1) $\phi(H)$ is a normal subgroup of G' if H is a normal subgroup of G.
(2) $\phi^{-1}(K')$ is a normal subgroup of G for a normal subgroup K' of G'.

Proof. As H is a subgroup of G, by previous theorem $\phi(H)$ is a subgroup of G'. To show $\phi(H)$ is a normal subgroup of G', let us take $x' \in G'$ and $h' \in \phi(H)$. Since ϕ is a onto homomorphism, $G' = \phi(G)$, so there exists an element $x \in G$ such that $\phi(x) = x'$. Also there exist an element $h \in H$ such that $\phi(h) = h'$. Now H is normal in G implies $xhx^{-1} \in H$ and hence $x' * h' * x'^{-1} = \phi(x) * \phi(h) * \phi(x)^{-1} = \phi(xhx^{-1}) \in \phi(H)$ and hence the theorem.
(2) Similar proof □

Definition 0.3. Let $(G,\cdot)$ and $(G',*)$ be two groups and $\phi : G \to G'$ be a homomorphism.Then *Kernel of* ϕ, a subset of G is defined as

$$\ker\phi = \{a \in G : \phi(a) = e_{G'}\}$$

Theorem 0.4. *Let G and G' be two groups and $\phi : G \to G'$ be a homomorphism. Then $\ker\phi$ is a normal subgrouop of G.*

Proof. Since $\phi(e_G) = e_{G'}$, $e_G \in \ker\phi$, hence $\ker\phi$ is non-empty. Let us take two elements $a, b \in \ker\phi$, then $\phi(ab^{-1}) = \phi(a)\phi(b)^{-1} = e_{G'}e_{G'} = e_{G'}$,hence $ab^{-1} \in \ker\phi$. This proves that $\ker\phi$ is a subgroup of G. To show this is normal in G, let us take $g \in G$ and $h \in \ker\phi$ then $\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g)^{-1} = \phi(g)e_{G'}\phi(g)^{-1} = e_{G'}$.Thus $ghg^{-1} \in \ker\phi$. Hence the theorem. □

Theorem 0.5. *Let $\phi : G \to G'$ be a homomorphism. Then ϕ is injective if and only if* $\ker \phi = \{e_G\}$

Proof. Let us assume ϕ is injective. Now $\phi(e_G) = e_{G'}$ implies e_G is the only preimage of e_G, as ϕ is injective. Hence $\ker \phi = \{e_G\}$.
Conversely let $\ker \phi = \{e_G\}$. To show ϕ is injective let $\phi(a) = \phi(b)$ fow two elements $a, b \in G$. Now $\phi(ab^{-1}) = \phi(a)\phi(b)^{-1} = \phi(a)\phi(a)^{-1} = e_{G'}$. So $ab^{-1} = e_G$ implies $a = b$. So ϕ is injective. □

Definition 0.6. Let G and G' be two groups and $\phi : G \to G'$ be a homomorphism. ϕ is said to be isomorphism if it is one-one and onto.

Note 0.7. Let $\phi : G \to G'$ be an epimorphism. Then ϕ is an isomorphism if and only if $\ker \phi = \{e_G\}$.

Note 0.8. Let $\phi : G \to G'$ be an isomorphism. Then $\phi^{-1} : G' \to G$ is also an isomorphism.

Note 0.9. Let $\phi : G \to G'$ be an isomorphism. Then

(1) $o(a) = o(\phi(a))$ for every $a \in G$.

(2) The set G and G' have the same cardinality.

Theorem 0.10. *Let $\phi : G \to G'$ be an isomorphism. Then*

(1) G' is commutative if and only if G is so.

(2) G' is cyclic if and only if G is so.

Proof. (1) Let us suppose G is commutative. To show G' is commutative let us take two elements a', b' from G'. As ϕ is an isomorphism there exists two elements $a, b \in G$ such that $\phi(a) = a'$ and $\phi(b) = b'$. Now $a'b' = \phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a) = b'a'$. Conversely as ϕ is an isomorphism , ϕ^{-1} is also an isomorphism. So by first part commutativeness of G' implies that G is commutative.

(2)Let us suppose G is cyclic and $G =< a >$ for some $a \in G$. We now show that $\phi(a)$ is a generator of G'. Let us take b' from G', so there exists an element $b \in G$ such that $\phi(b) = b'$. Now $G =< a >$, so there exists an integer r such that $b = a^r$, then $b' = \phi(b) = \phi(a^r) = \phi(a)^r$. Hence G' is cyclic. Conversely,as ϕ is an isomorphism , ϕ^{-1} is also an isomorphism. So by first part cyclicness of G' implies that G is cyclic. □