VIVEKANANDA COLLEGE
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NAAC ACCREDITED 'A' GRADE

Topic: Power Series

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MATHEMATICS

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POWER SERIES

(Properties of Power Series)

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Theorem 0.6. *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with radius of convergence* $R > 0$. *Then the series is uniformly convergent on* $[-s, s]$, *for* $0 < s < R$.

Proof. Let $f_n(x) = a_n x^n$, $n > 0$. As R is the radius of convergent of the power series, so the series is absolutely convergent for all real x satisfying $|x| < R$. Now $0 < s < R$ implies the power series is absolutely convergent for any x with $0 < |x| \leq s$. Thus $\sum_{n=0}^{\infty} |a_n s^n|$ is covergent.

Again $|f_n(x)| = |a_n x^n| \leq |a_n| s^n$ for $|x| \leq s$. If we take $M_n = |a_n| s^n$, then $\sum_{n=0}^{\infty} M_n$ is a convergent series of positive terms for and $|f_n(x)| \leq M_n$ for all $|x| \leq s$, for all $n \in \mathbb{N}$. Hence by Weierstrass' M Test the series is uniformly convergent on $[-s, s]$. □

Note 0.7. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$.Then the series is uniformly convergent on $[-R+\epsilon, R-\epsilon]$, for any small $\epsilon > 0$ with $R - \epsilon > 0$.

Note 0.8. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$.Then the series is uniformly convergent on any closed bounded interval $[a, b] \subset (-R, R)$.

Theorem 0.9. *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with radius of convergence* $R > 0$ *and* $f(x)$ *be the sum of the series on* $(-R, R)$*. Then* f *is continuous.*

Proof. Same as we have done in previous chapter. $\square$

Theorem 0.10. *A power series can be integrated term by term on any closed bounded interval containded within the interval of convergence.*

Proof. Same as we have done in previous chapter. $\square$

Theorem 0.11. *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with radius of convergence* $R > 0$. *Then the radius of convergence of the power series* $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ *, obtained by term by term integration is also* R.

Proof. We know that $\frac{1}{R} = \overline{\lim}|a_n|^{\frac{1}{n}}$. Let R' be the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$. The nth term of the power series is $\frac{a_{n-1}}{n} x^n$.

So $\frac{1}{R'} = \overline{\lim}|\frac{a_{n-1}}{n}|^{\frac{1}{n}} = \overline{\lim}\frac{\{|a_{n-1}|^{\frac{1}{n-1}}\}^{\frac{n-1}{n}}}{n^{\frac{1}{n}}} = \frac{1}{R}$, as $\overline{\lim}|a_{n-1}|^{\frac{1}{n-1}} = \frac{1}{R}$ and $\overline{\lim}\frac{n-1}{n} = 1$, $\overline{\lim} n^{\frac{1}{n}} = 1$. □

Theorem 0.12. *Let $R > 0$ be the radius of convergence of the power series of* $\sum_{n=0}^{\infty} a_n x^n$. *Then the radius of convergence of the power series* $\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$ *, obtained by term by termndifferentiation is also R.*

Proof. We know that $\frac{1}{R} = \overline{\lim}|a_n|^{\frac{1}{n}}$. Let R' be the radius of convergence of the power series $\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$.

Then $\frac{1}{R'} = \overline{\lim}|(n+1)a_{n+1}|^{\frac{1}{n}} = \overline{\lim}\{(n+1)^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}\{|a_{n+1}|^{\frac{1}{n+1}}\}^{\frac{n+1}{n}} = \frac{1}{R}$ as $\lim(n+1)^{\frac{1}{n+1}} = 1$ and $\lim \frac{n+1}{n} = 1$ and $= \overline{\lim}|a_{n+1}|^{\frac{1}{n+1}} = \frac{1}{R}$. Hence $R = R'$. □