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NAAC ACCREDITED ‘A’ GRADE

Topic: Power Series

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Power Series

(Taylor's series form of sum function

&

Uniqueness of power series having sum function)

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Note 0.13. A power series can be differentiated term by term within the interval of convergence.

Theorem 0.14. *Let $R > 0$ be the radius of convergence of the power series of $\sum_{n=0}^{\infty} a_n x^n$ and $f(x)$ be the sum of the series on $(-R, R)$. Then $f^k(0) = k!a_k$, $k = 0, 1, 2, \ldots$.*

Proof. $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ on $(-R, R)$.

$\implies f(0) = a_0$ so $f^0(0) = 0!a_0$.

Differentiating term by term, we get $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$ on $(-R, R)$.

$\implies f'(x) = a_1$ so $f'(0) = 1!a_1$.

Again differentiating term by term we get $f''(x) = 1.2a_2 + 2.3x + \ldots$ on $(-R, R)$

$\implies f''(0) = 2!a_2$.

Continuing the process we obtain $f^k(0) = k!a_k$ or $a_k = \frac{f^k(0)}{k!}$, $k = 0, 1, 2, \ldots$ □

Definition 0.15. If a function f defined on some neighbourhood of 0 has derivatives of all orders on that neighbourhood then the series $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$ is called the *Taylor's series* of f about the point 0.

Remark 0.16. From the above theorem we know that the power series takes of the form $\sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n$, which is the Taylor's series of the sum function about the origin.Thus every power series is a Taylor's series of its sum function.

Now the question aries if a function f having derivatives of all orders on some neighbourhood of 0 and the Taylor's series $\sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n$ be constructed does this power series will have f as is sum function on hat neighbourhood?

The answer is "NO".

For example let us take the function $f(x) = e^{-\frac{1}{x^2}}, x \neq 0$ with $f(0) = 0$.

Then $f^n(0) = 0$ for $n = 0, 1, 2, \ldots$ The Taylor series of f about 0 is $0 + 0 + 0 + \ldots$ and this converges to 0 not to f.

Theorem 0.17. *UNIQUENESS THEOREM*

If two power series takes the form $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} b_n x^n$ *converge in the same interval* $(-R, R)$ *to the same function* f, *then* $a_n = b_n$, $\forall n = 1, 2, \ldots$

Proof. We have, $a_0 + a_1 x + a_2 x^2 + \cdots = f(x) = b_0 + b_1 x + b_2 x^2 \cdots$ on $(-R, R)$.

Putting $x = 0, a_0 = f(0) = b_0$

Differentiating term by term both the series on $(-R, R)$.

$a_1 + 2a_2 x + 3a_3 x^2 + \cdots = f'(x) = b_1 + 2b_2 x + 3b_3 x^2 + \cdots$ on $(-R, R)$,

Putting $x = 0, a_1 = f'(0) = b_1$. Continuing this process we obtain $a_k = f^k(0) = b_k$, $\forall k$.

$a_n = b_n$, $\forall n = 0, 1, 2, \cdots$. □

Example 0.18. Find the exact interval of convergence of the power series $1-\frac{x^2}{3}+\frac{x^4}{5}-\cdots$.

Solution: Let us take $y = x^2$, then the series becomes $\sum_{n=0}^{\infty} a_n y^n$, where $a_0 = 1$ and $a_n = \frac{(-1)^n}{2n+1}$, $n > 1$.

Now $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{2n+1}{2n+3} = 1.$

So by ratio test radius of convergence of the power series $\sum_{n=0}^{\infty} a_n y^n$ is 1.

Therefore $\sum_{n=0}^{\infty} a_n y^n$ is absolutely convergence for $|y| < 1 \implies$ for $|x^2| < 1$.

$\sum_{n=0}^{\infty} a_n x^{2n}$ converges for $|x| < 1$.

Again for $|x| = 1$ the series becomes $1 - \frac{1}{3} + \frac{1}{5} - \cdots$, convergent by Leibnitz test .

Therefore $1 - \frac{x^2}{3} + \frac{x^4}{5} - \cdots$ converges in the interval $[-1, 1]$

Example 0.19. Prove that the power series $\sum_{n=0}^{\infty} \frac{e^n}{(2n)!} x^n$ is convergent everywhere.

Solution: Here $a_n = \frac{e^n}{(2n)!}$ for $n = 0, 1, 2, \ldots$

Now $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{e^{n+1}}{(2n+2)!} \frac{2n!}{e^n} = \lim_{n\to\infty} \frac{e}{(2n+2)(2n+1)} = 0$

So by ratio test the radius of convergent of the power series is ∞. Thus the series is everywhere convergent.