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Chapter Nine

The energy principle

and energy conservation

KEY FEATURES

The key features of this chapter are the **energy principle** for a multi-particle system, the **potential energies** arising from **external** and **internal** forces, and **energy conservation**.

This is the first of three chapters in which we study the mechanics of **multi-particle systems**. This is an important development which greatly increases the range of problems that we can solve. In particular, multi-particle mechanics is needed to solve problems involving the rotation of rigid bodies.

The chapter begins by obtaining the **energy principle** for a multi-particle system. This is the first of the three great principles of multi-particle mechanics* that apply to *every* mechanical system without restriction. We then show that, under appropriate conditions, the total energy of the system is conserved. We apply this **energy conservation** principle to a wide variety of systems. When the system has just one degree of freedom, the energy conservation equation is sufficient to determine the whole motion.

9.1 CONFIGURATIONS AND DEGREES OF FREEDOM

A **multi-particle system** $\mathcal{S}$ may consist of any number of particles P_1, P_2, ..., P_N, with masses m_1, m_2 ..., m_N respectively.† A possible 'position' of the system is called a **configuration**. More precisely, if the particles P_1, P_2, ..., P_N of a system have position vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, ..., $\boldsymbol{r}_N$, then any *geometrically possible* set of values for the position vectors $\{\boldsymbol{r}_i\}$ is a configuration of the system.

If the system is **unconstrained**, then each particle can take up any position in space (independently of the others) and all choices of the $\{\boldsymbol{r}_i\}$ are possible. This would be the case, for instance, if the particles of $\mathcal{S}$ were moving freely under their mutual gravitation. On the other hand, when **constraints** are present, the $\{\boldsymbol{r}_i\}$ are restricted. Suppose for instance that the particles P_1 and P_2 are connected by a light rigid rod of length a.

* The other two are the *linear momentum* and *angular momentum* principles.

† To save space, we will usually express this by saying that $\mathcal{S}$ is the system of particles $\{P_i\}$ with masses $\{m_i\}$, the range of the index number i being understood to be $1 \le i \le N$.

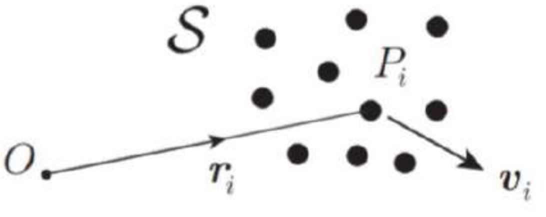

FIGURE 9.1 The multi-particle system $\mathcal{S}$ consists of N particles $P_1, P_2, \ldots, P_N$, of which the typical particle P_i is labelled. The particle P_i has mass m_i, position vector $\boldsymbol{r}_i$, and velocity $\boldsymbol{v}_i$.

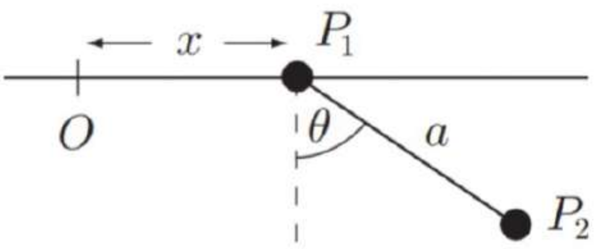

FIGURE 9.2 The generalised coordinates x and θ are sufficient to specify the configuration of this two-particle system in planar motion.

This imposes the geometrical restriction $|\boldsymbol{r}_1 - \boldsymbol{r}_2| = a$ so that not all choices of the $\{\boldsymbol{r}_i\}$ are then possible. This difference is reflected in the number of scalar variables needed to specify the configuration of $\mathcal{S}$. In the unconstrained case, all of the position vectors $\{\boldsymbol{r}_i\}$ must be specified separately. Since each of these vectors may be specified by three Cartesian coordinates, it follows that a total of $3N$ *scalar variables are needed to specify the configuration of an unconstrained N-particle system*. When constraints are present, this number is reduced, often dramatically so.

For example, consider the system shown in Figure 9.2, which consists of two particles P_1 and P_2 connected by a light rigid rod of length a. The particle P_1 is also constrained to move along a fixed horizontal rail and the whole system moves in the vertical plane through the rail. The *two* scalar variables x and θ shown are sufficient to specify the configuration of this system. This contrasts with the *six* scalar variables that would be needed if the two particles were in unconstrained motion. The variables x and θ are said to be a set of **generalised coordinates** for this system.* Other choices for the generalised coordinates could be made, but the *number* of generalised coordinates needed is always the same.

Definition 9.1 ***Degrees of freedom*** *The* ***number*** *of generalised coordinates needed to specify the configuration of a system $\mathcal{S}$ is called the number of* ***degrees of freedom*** *of $\mathcal{S}$.*

Importance of degrees of freedom

The number of degrees of freedom of a system is important because it is equal to the *number of equations that are needed to determine the motion of the system*. For example,

* Besides being sufficient to specify the configuration of the system, the generalised coordinates are also required to be *independent*, that is, there must be no functional relation between them. The coordinates x, θ in Figure 9.2 are certainly independent variables. If the coordinates were connected by a functional relation, they would not all be needed and one of them could be discarded.

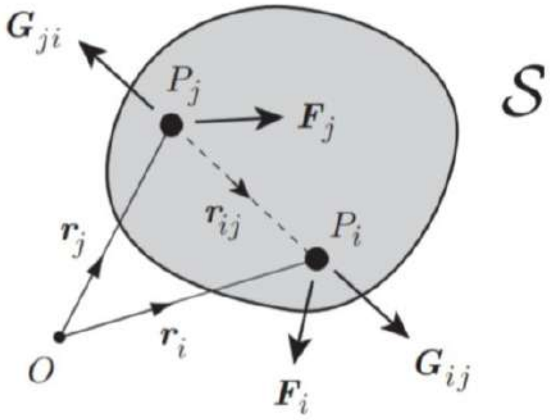

FIGURE 9.3 The multi-particle $\mathcal{S}$ consists of N particles $P_1, P_2, \ldots, P_N$, of which the typical particles P_i and P_j are shown explicitly. The force $\boldsymbol{F}_i$ is the *external* force acting on P_i and the force $\boldsymbol{G}_{ij}$ is the *internal* force exerted on P_i by the particle P_j.

the system shown in Figure 9.2 has *two* degrees of freedom and so needs *two* equations to determine the motion completely.

Example 9.1 *Degrees of freedom*

Find the number of degrees of freedom of the following mechanical systems: (i) the simple pendulum (moving in a vertical plane), (ii) a door swinging on its hinges, (iii) a bar of soap (a particle) sliding on the inside of a hemispherical basin, (iv) a rigid rod sliding on a flat table, (v) four rigid rods flexibly jointed to form a quadrilateral which can slide on a flat table.

Solution

(i) 1 (ii) 1 (iii) 2 (iv) 3 (v) 4. ■

9.2 THE ENERGY PRINCIPLE FOR A SYSTEM

Let $\mathcal{S}$ be a system of N particles $\{P_i\}$, as shown in Figure 9.3. We classify the forces acting on the particles of $\mathcal{S}$ as being external or internal. **External forces** are those originating from *outside* $\mathcal{S}$. (In the case of a single particle, these are the only forces that act.) Uniform gravity is an example of an external force. However, in multi-particle systems, the particles are also subject to their own *mutual interactions*, that is, the forces that they exert upon each other. These mutual interactions are called the **internal forces** acting on $\mathcal{S}$. The situation is shown in Figure 9.3. $\boldsymbol{F}_i$ is the external force acting on the particle P_i, while $\boldsymbol{G}_{ij}$ is the internal force exerted on P_i by the particle P_j. By the Third Law, the force $\boldsymbol{G}_{ji}$ that P_i exerts on P_j must be equal and opposite to the force $\boldsymbol{G}_{ij}$, and both forces must be parallel to the straight line joining P_i and P_j. In short, the $\{\boldsymbol{G}_{ij}\}$ must satisfy

$$\boldsymbol{G}_{ji} = -\boldsymbol{G}_{ij}, \qquad \text{and} \qquad \boldsymbol{G}_{ij} \parallel (\boldsymbol{r}_i - \boldsymbol{r}_j). \tag{9.1}$$

To obtain the energy principle for the system $\mathcal{S}$, we proceed in the same way as we did for a single particle in section 6.1. The equation of motion for the particle P_i is*

$$m_i \frac{d\boldsymbol{v}_i}{dt} = \boldsymbol{F}_i + \sum_{j=1}^{N} \boldsymbol{G}_{ij}, \tag{9.2}$$

where $\boldsymbol{v}_i$ is the velocity of P_i at time t. On taking the scalar product of both sides of equation (9.2) with $\boldsymbol{v}_i$ and then *summing* the result over all the particles ($1 \le i \le N$), we obtain

$$\frac{dT}{dt} = \sum_{i=1}^{N} \left\{ \boldsymbol{F}_i + \sum_{j=1}^{N} \boldsymbol{G}_{ij} \right\} \cdot \boldsymbol{v}_i, \tag{9.3}$$

where

$$T = \sum_{i=1}^{N} \tfrac{1}{2} m_i |\boldsymbol{v}_i|^2,$$

the **total kinetic energy** of the whole system $\mathcal{S}$. Suppose that, in the time interval $[t_{\mathcal{A}}, t_{\mathcal{B}}]$, the system $\mathcal{S}$ moves from configuration $\mathcal{A}$ to configuration $\mathcal{B}$. On integrating equation (9.3) with respect to t over the time interval $[t_{\mathcal{A}}, t_{\mathcal{B}}]$ we obtain

$$T_{\mathcal{B}} - T_{\mathcal{A}} = \sum_{i=1}^{N} \int_{t_{\mathcal{A}}}^{t_{\mathcal{B}}} \boldsymbol{F}_i \cdot \boldsymbol{v}_i \, dt + \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{t_{\mathcal{A}}}^{t_{\mathcal{B}}} \boldsymbol{G}_{ij} \cdot \boldsymbol{v}_i \, dt \tag{9.4}$$

where $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$ are the kinetic energies of the system $\mathcal{S}$ at times $t_{\mathcal{A}}$ and $t_{\mathcal{B}}$ respectively. This is the **energy principle** for a multi-particle system moving under the external forces $\{\boldsymbol{F}_i\}$ and internal forces $\{\boldsymbol{G}_{ij}\}$. This impressive looking result can be stated quite simply as follows:

Energy principle for a multi-particle system

In any motion of a system, the increase in the total kinetic energy of the system in a given time interval is equal to the total work done by all the external and internal forces during this time interval.

* The summation over j in equation (9.2) contains the term $\boldsymbol{G}_{ii}$ which corresponds to the force that the particle P_i exerts upon *itself*. Since such a force is not actually present, we should really say that the summation is over the range $1 \le j \le N$ with $j \ne i$. Since this would make the formulae look messy, we adopt the device of regarding the terms $\boldsymbol{G}_{11}, \boldsymbol{G}_{22}, \ldots, \boldsymbol{G}_{NN}$ (which do not actually exist) as being zero.

9.3 ENERGY CONSERVATION FOR A SYSTEM

In order to develop an energy conservation principle, we need to write the right side of the energy principle (9.4) in the form $V(\mathcal{A}) - V(\mathcal{B})$, where V is the potential energy function for the *whole system*. We first consider *unconstrained* systems.

Unconstrained systems

When the system is unconstrained, all the forces that act on the system are specified directly. We will assume that the **external forces** $\boldsymbol{F}_i$ are *conservative fields*. In this case $\boldsymbol{F}_i = -\operatorname{grad}\phi_i$, where ϕ_i is the potential energy function of the field $\boldsymbol{F}_i$. Then the total work done by the external forces can be written

$$\sum_{i=1}^{N}\int_{t_A}^{t_B} \boldsymbol{F}_i \cdot \boldsymbol{v}_i \, dt = \sum_{i=1}^{N}\left(\phi_i(\boldsymbol{r}_A) - \phi_i(\boldsymbol{r}_B)\right) = \Phi(\mathcal{A}) - \Phi(\mathcal{B}),$$

where

$$\Phi(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) = \phi_1(\boldsymbol{r}_1) + \phi_2(\boldsymbol{r}_2) + \cdots + \phi_N(\boldsymbol{r}_N)$$

is the potential energy of $\mathcal{S}$ arising from the **external** forces.

Example 9.2 *Potential energy under uniform gravity*

Find the potential energy Φ when the external forces on $\mathcal{S}$ arise from uniform gravity.

Solution

Under uniform gravity, the force $\boldsymbol{F}_i$ exerted on particle P_i is $\boldsymbol{F}_i = -m_i g\boldsymbol{k}$, where the unit vector $\boldsymbol{k}$ points vertically upwards. This conservative field has potential energy $\phi_i = m_i g z_i$, where z_i is the z-coordinate of P_i. The total potential energy of $\mathcal{S}$ due to uniform gravity is therefore

$$\Phi = m_1 g z_1 + m_2 g z_2 + \cdots + m_N g z_N.$$

On using the definition of centre of mass given in section 3.5, this can be written in the alternative form

$$\Phi = MgZ,$$

where M is the total mass of $\mathcal{S}$, and Z is the z-cordinate of the centre of mass of $\mathcal{S}$. Thus *the potential energy of any system due to uniform gravity is the same as if all its mass were concentrated at its centre of mass.* ■

We now need to make a similar transformation to show that the work done by the **internal forces** can be written in the form $\Psi(\mathcal{A}) - \Psi(\mathcal{B})$, where Ψ is the internal potential energy. The argument is as follows:

We know from the Third Law that the $\{\boldsymbol{G}_{ij}\}$ satisfy the conditions (9.1), but a little more must be assumed. We further assume that the *magnitude* of $\boldsymbol{G}_{ij}$ depends only on r_{ij}, the distance between P_i and

P_j.* Internal forces that satisfy this conditions will be called **conservative**; mutual gravitation forces are a typical example. Hence, when the internal forces are conservative, $\boldsymbol{G}_{ij}$ must have the form

$$\boldsymbol{G}_{ij} = h_{ij}(r_{ij})\,\widehat{\boldsymbol{r}}_{ij} \tag{9.5}$$

where (see Figure 9.3)

$$\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j \qquad r_{ij} = |\boldsymbol{r}_{ij}| \qquad \widehat{\boldsymbol{r}}_{ij} = \boldsymbol{r}_{ij}/r_{ij}. \tag{9.6}$$

Note that h_{ij} is the *repulsive* force that the particles P_i and P_j exert upon each other.

Consider now the rate of working of the *pair* of forces $\boldsymbol{G}_{ij}$ and $\boldsymbol{G}_{ji}$. This is

$$\begin{aligned}\boldsymbol{G}_{ij}\cdot\boldsymbol{v}_i + \boldsymbol{G}_{ji}\cdot\boldsymbol{v}_j = \boldsymbol{G}_{ij}\cdot\left(\boldsymbol{v}_i - \boldsymbol{v}_j\right) &= h_{ij}(r_{ij})\,\widehat{\boldsymbol{r}}_{ij}\cdot\frac{d\boldsymbol{r}_{ij}}{dt} = \left(\frac{h_{ij}(r_{ij})}{r_{ij}}\right)\boldsymbol{r}_{ij}\cdot\frac{d\boldsymbol{r}_{ij}}{dt}\\ &= h_{ij}(r_{ij})\,\frac{dr_{ij}}{dt},\end{aligned}$$

on using equations (9.1), (9.6) and the identity $\boldsymbol{r}_{ij}\cdot\dot{\boldsymbol{r}}_{ij} = r_{ij}\dot{r}_{ij}$. The total work done by the forces $\boldsymbol{G}_{ij}$ and $\boldsymbol{G}_{ji}$ during the time interval $[t_{\mathcal{A}}, t_{\mathcal{B}}]$ is therefore

$$\int_{t_{\mathcal{A}}}^{t_{\mathcal{B}}} h_{ij}(r_{ij})\,\frac{dr_{ij}}{dt}\,dt = \int_{r_{ij}(\mathcal{A})}^{r_{ij}(\mathcal{B})} h_{ij}(r_{ij})\,dr_{ij} = H_{ij}(r_{ij}(\mathcal{A})) - H_{ij}(r_{ij}(\mathcal{B})),$$

where H_{ij} is the indefinite integral of $-h_{ij}$. The function $H_{ij}(r_{ij})$ is called the **mutual potential energy** of the particles P_i and P_j.

It follows that the total work done by all the internal forces in the time interval $[t_{\mathcal{A}}, t_{\mathcal{B}}]$ can be written in the form

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{t_{\mathcal{A}}}^{t_{\mathcal{B}}} \boldsymbol{G}_{ij}\cdot\boldsymbol{v}_i\,dt = \Psi(\mathcal{A}) - \Psi(\mathcal{B}),$$

where

$$\Psi(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) = \sum_{i=1}^{N}\sum_{j=1}^{i-1} H_{ij}(r_{ij})$$

is the **potential energy** of $\mathcal{S}$ arising from the **internal** forces. This potential energy is just the sum of the mutual potential energies of all pairs of particles.

Example 9.3 *Internal energy of three charged particles*

Three particles P_1, P_2, P_3 carry electric charges e_1, e_2, e_3 respectively. Find the internal potential energy Ψ.

Solution

In cgs/electrostatic units, the particles P_1 and P_2 repel each other with the force $h_{12}(r_{12}) = e_1 e_2/(r_{12})^2$, where r_{12} is the distance between P_1 and P_2. Their mutual potential energy is therefore

$$H_{12} = -\int h_{12}(r_{12})\,dr_{12} = -\int \frac{e_1 e_2}{(r_{12})^2}\,dr_{12} = \frac{e_1 e_2}{r_{12}}.$$

* This is equivalent to the very reasonable assumptions that the magnitude of $\boldsymbol{G}_{ij}$ is invariant under spatial translations and rotations of each pair of particles P_i and P_j, and is independent of the time.

The **internal potential energy** of the whole system is therefore

$$\Psi = \frac{e_1 e_2}{r_{12}} + \frac{e_1 e_3}{r_{13}} + \frac{e_2 e_3}{r_{23}}. \blacksquare$$

On combining the above results, the energy principle (9.4) can be written

$$T_{\mathcal{B}} - T_{\mathcal{A}} = V(\mathcal{A}) - V(\mathcal{B}),$$

where $V = \Phi + \Psi$ is the **total potential energy** of the system $\mathcal{S}$. This is equivalent to the **energy conservation** formula

$$\boxed{T + V = E} \qquad (9.7)$$

where E is the total energy of the system.This result can be summarised as follows:

Energy conservation for an unconstrained system

When both the external and internal forces acting on a system are *conservative*, the sum of its kinetic and potential energies* remains constant in the motion.

Example 9.4 *A star with two planets*

A star of very large mass M is orbited by two planets P_1 and P_2 of masses m_1 and m_2. Find the energy conservation equation for this system.

Solution

Since the mass of the star is supposed to be very much larger than the planetary masses, we will neglect its motion and suppose that it is fixed at the origin O. We then have a *two-particle* problem in which the planets move under the (external) gravitational attraction of the star and their (internal) mutual gravitational interaction. This is an unconstrained system.

The total potential energy arising from **external forces** is then

$$\Phi = -\frac{Mm_1G}{r_1} - \frac{Mm_2G}{r_2},$$

where r_1,r_2 are the distances OP_1, OP_2.

The particles P_1 and P_2 repel each other with the force $h_{12}(r_{12}) = -m_1m_2G/(r_{12})^2$, where r_{12} is the distance between P_1 and P_2. Their **mutual potential energy** is therefore

$$H_{12} = -\int h_{12}(r_{12})\, dr_{12} = \int \frac{m_1m_2G}{(r_{12})^2}\, dr_{12} = -\frac{m_1m_2G}{r_{12}},$$

and this is the only contribution to the internal potential energy Ψ.

* The potential energy is the total of the potential energies arising from both the external and internal forces.

Since the system is unconstrained and the external and internal forces are conservative, energy conservation applies. The **energy conservation equation** for the system is

$$\tfrac{1}{2}m_1\,|\boldsymbol{v}_1|^2 + \tfrac{1}{2}m_2\,|\boldsymbol{v}_2|^2 - MG\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) - \frac{m_1 m_2 G}{r_{12}} = E,$$

where $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are the velocities of the planets P_1, P_2, and E is the constant total energy. The value of E is determined from the initial conditions.

Since this system has six degrees of freedom (four if the motions are confined to a plane through O), the energy conservation equation is by no means sufficient to determine the motion! ■

Question ***Can a planet escape?***

If the initial conditions are such that $E < 0$, is it possible for a planet to escape to infinity?

Answer

If $E < 0$, then it is certainly not possible for *both* planets to escape to infinity, since the total energy would then be positive. However, the escape of *one* planet is not prohibited by energy conservation. This does not mean however that such an escape will actually happen.

Constrained systems

When a system is subject to constraints, not all the forces that act on the system are specified. This is because constraints are enforced by **constraint forces** that are not part of the specification of the problem; all we know is that their *effect* is to enforce the given constraints. The work done by constraint forces cannot generally be calculated (or expressed in terms of a potential energy) and we are restricted to those *systems for which the total work done by the constraint forces happens to be zero.**

The constraint forces acting on the system may be **external** (for example, when a particle of the system is constrained to remain at rest), or **internal** (for example, when two particles of the system are constrained to remain the same distance apart).

A The list of **external** constraint forces that do no work is the same as that given in Section 6.5 for single particle motion.

B The most important result regarding **internal** constraint forces that do no work is this: *The total work done by any pair of mutual interaction forces is zero when the particles on which they act are constrained to remain a fixed distance apart.* The proof is as follows:

Suppose two particles P_i and P_j are constrained to remain a fixed distance apart and that their mutual interaction forces are $\boldsymbol{G}_{ij}$ and $\boldsymbol{G}_{ji}$ (see Figure 9.3). Since the distance between P_i and P_j is constant, it follows that $(\boldsymbol{r}_i - \boldsymbol{r}_j)\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)$ is constant, which, on differentiating with respect

* *Individual* constraint forces may do work.

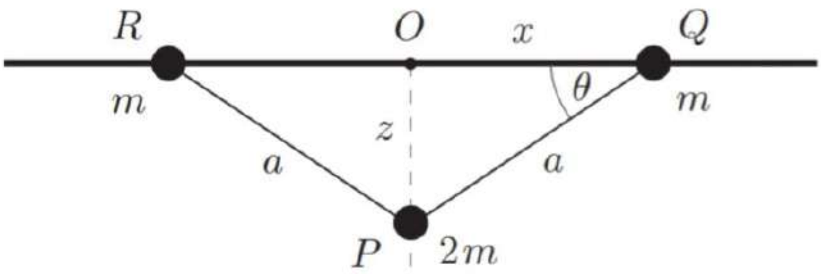

FIGURE 9.4 The particles Q and R slide along a smooth horizontal rail while the particle P moves vertically.

to t, gives

$$(\boldsymbol{r}_i - \boldsymbol{r}_j) \cdot (\boldsymbol{v}_i - \boldsymbol{v}_j) = 0.$$

Thus the vector $(\boldsymbol{v}_i - \boldsymbol{v}_j)$ must be *perpendicular* to the straight line joining P_i and P_j. Hence, the rate of working of the *two* forces $\boldsymbol{G}_{ij}$ and $\boldsymbol{G}_{ji}$ is

$$\boldsymbol{G}_{ij} \cdot \boldsymbol{v}_i + \boldsymbol{G}_{ji} \cdot \boldsymbol{v}_j = \boldsymbol{G}_{ij} \cdot \left(\boldsymbol{v}_i - \cdot \boldsymbol{v}_j\right) = 0,$$

since $\boldsymbol{G}_{ij}$ is known to be *parallel* to the straight line joining P_i and P_j. Thus the internal constraint forces $\boldsymbol{G}_{ij}$ and $\boldsymbol{G}_{ji}$ do no work *in total*.

It follows, for example, that the two tension forces exerted by a light inextensible string do no work in total. It further follows that the *internal forces that enforce rigidity in a rigid body do no work in total*. This important result allows us to solve rigid body problems by energy methods.

Our result for constrained systems can be summarised as follows:

Energy conservation for a constrained system

When the specified external and internal forces acting on a system are *conservative*, and the constraint forces *do no work in total*, the sum of the kinetic and potential energies of the system remains constant in the motion.

Example 9.5 ***A constrained three-particle system***

Figure 9.4 shows a ball P of mass $2m$ suspended by light inextensible strings of length a from two sliders Q and R, each of mass m, which can move on a smooth horizontal rail. The system moves symmetrically so that O, the mid-point of Q and R, remains fixed and P moves on the downward vertical through O. Initially the system is released from rest with the three particles in a straight line and with the strings taut. Find the energy conservation equation for the system.

Solution

This is a system with one degree of freedom and we take the angle θ as the generalised coordinate. Let z and x be the displacements of the particles P and Q from the fixed point O. Then, in terms of the generalised coordinate θ, $x = a\cos\theta$ and $z = a\sin\theta$. Differentiating these formulae with respect to t then gives

$$\dot{x} = -(a\sin\theta)\dot{\theta}, \qquad \dot{z} = (a\cos\theta)\dot{\theta}.$$

Hence the total **kinetic energy** of the system is given by

$$T = \tfrac{1}{2}(2m)\dot{z}^2 + \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{x}^2 = ma^2\dot{\theta}^2.$$

The only contribution to the **potential energy** comes from uniform gravity, so that

$$V = -(2m)gz + 0 + 0 = -2mga\sin\theta,$$

where we have taken the zero level of potential energy to be at the rail.

We must now show that the constraint forces do no work. The reactions exerted by the smooth rail on the particles Q and R are perpendicular to the rail and therefore perpendicular to the velocities of Q and R; these reactions therefore do no work. Also, the tension forces exerted by the inextensible strings do no work in total. Hence, the **constraint forces** do no work in total.

Energy conservation therefore applies in the form

$$ma^2\dot{\theta}^2 - 2mga\sin\theta = E.$$

From the initial conditions $\theta = \dot{\theta} = 0$ when $t = 0$, it follows that $E = 0$. The **energy conservation equation** for the system is therefore

$$\dot{\theta}^2 - \frac{2g}{a}\sin\theta = 0. \blacksquare$$

Question ***When do the sliders collide?***

Find the time that elapses before the sliders collide.

Answer

Since this system has only one degree of freedom, the motion can be found from energy conservation alone. From the energy conservation equation, it follows that

$$\frac{d\theta}{dt} = \pm\left(\frac{2g}{a}\right)^{1/2}(\sin\theta)^{1/2},$$

and, since θ is an *increasing* function of t, we take the *positive* sign. This equation is a first order separable ODE.

Since the sliders collide when $\theta = \pi/2$, the time τ that elapses is given by

$$\tau = \left(\frac{a}{2g}\right)^{1/2}\int_0^{\pi/2}\frac{d\theta}{(\sin\theta)^{1/2}} \approx 1.85\left(\frac{a}{g}\right)^{1/2}. \blacksquare$$

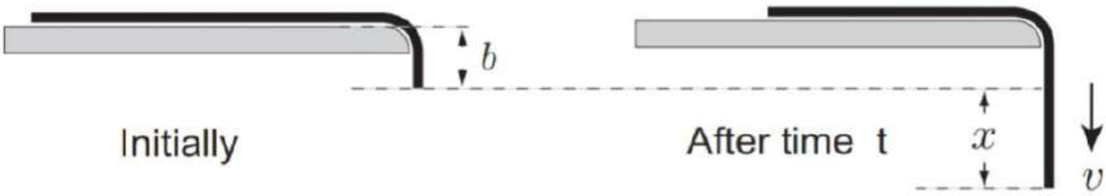

FIGURE 9.5 A uniform rope is released from rest hanging over the edge of a smooth table (left). After time t it has displacement x (right).

Example 9.6 *Rope sliding off a table*

A uniform inextensible rope of mass M and length a is released from rest hanging over the edge of a smooth horizontal table, as shown in Figure 9.5. Find the speed of the rope when it has the displacement x shown.

Solution

A rope is a continuous distribution of mass, unlike the discrete masses that appear in our theory. We regard the rope as being represented by a *light* inextensible string of length a with N particles, each of mass M/N, attached to the string at equally spaced intervals along its length. When N is very large, we expect this discrete set of masses to approximate the behaviour of the rope.

Since each particle of the rope has the same *speed* $v\,(=\dot{x})$, the total **kinetic energy** of the rope is simply

$$T = \tfrac{1}{2}Mv^2.$$

The only contribution to the **potential energy** comes from uniform gravity. If we take the reference state for V to be the initial configuration (Figure 9.5 (left)), then the potential energy in the displaced configuration (right) is the same *as if* a length x of the rope lying on the table were cut off and this piece were then suspended from the hanging end. In the continuous limit (that is, as $N \to \infty$), this piece of rope has mass Mx/a and its centre of mass is lowered a distance $b + (x/2)$ by this operation. The potential energy of the rope in the displaced configuration is therefore

$$V = -\left(\frac{Mx}{a}\right) g \left(b + \tfrac{1}{2}x\right).$$

We must now show that the constraint forces do no work. The reactions exerted by the smooth table on the particles of the rope are always perpendicular to the velocities of these particles; these reactions therefore do no work. Also, the tension forces exerted by each segment of the inextensible string (connecting adjacent particles of the rope) do no work in total. Hence, the **constraint forces** do no work in total.

Energy conservation therefore applies in the form

$$\tfrac{1}{2}Mv^2 - \left(\frac{Mx}{a}\right) g \left(b + \tfrac{1}{2}x\right) = E.$$

The initial condition $v = 0$ when $x = 0$ implies that $E = 0$. The energy equation for the rope is therefore

$$v^2 = \frac{g}{a} x(x + 2b).$$

This gives the **speed** of the rope when it has displacement x. This formula holds while there is still some rope left on the the table *top*. ■

Note. In the above solution we have assumed that the rope follows the contour of the table edge and then falls vertically. However, it can be shown that this *cannot* be true when the rope is close to leaving the table. What actually happens is that the end of the rope overshoots the table edge. This is a tricky point which we will not investigate further.

Question *Displacement at time t*

Find the displacement of the rope at time t.

Answer

Since this system has only one degree of freedom, the motion can be found from energy conservation alone. From the energy conservation equation, it follows that

$$\frac{dx}{dt} = \pm n\, x^{1/2}(x + 2b)^{1/2},$$

where $n^2 = g/a$. Since x is an *increasing* function of t, we take the *positive* sign. This equation is a first order separable ODE.

It follows that

$$\begin{aligned} nt &= \int \frac{dx}{x^{1/2}(x + 2b)^{1/2}} \\ &= 2 \sinh^{-1}\left(\frac{x}{2b}\right)^{1/2} + C, \end{aligned}$$

on using the substitution $x = 2b \sinh^2 w$. The initial condition $x = 0$ when $t = 0$ implies that $C = 0$ and, after some simplification, we obtain

$$x = b(\cosh nt - 1)$$

as the **displacement** of the rope after time t. As before, this formula holds while there is still some rope left on the the table top. ■

Example 9.7 *Stability of a plank on a log*

A uniform thin rigid plank is placed on top of a rough circular log and can roll without slipping. Show that the equilibrium position, in which the plank rests symmetrically on top of the log, is stable.

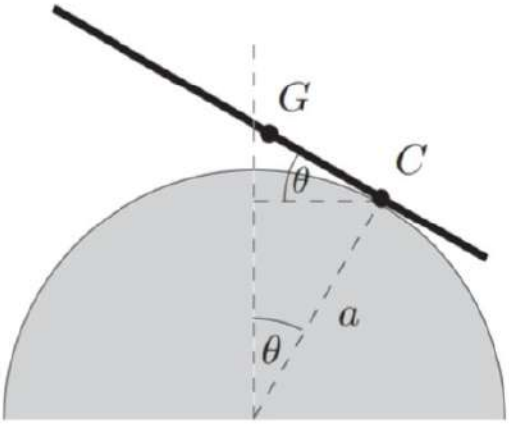

FIGURE 9.6 A thin uniform plank is placed symmetrically on top of a fixed rough circular log. Is the equilibrium position of the plank stable?

Solution

Suppose that the plank is disturbed from its equilibrium position and is tilted by an angle θ as shown in Figure 9.6. The plank is known to *roll* on the log, which means that the distance GC from the centre G of the plank to the contact point C must always be equal to the arc length of the log that has been traversed. If the radius of the log is a, then this arc length is $a\theta$.

We are not yet able to calculate the **kinetic energy** of the plank in terms of the coordinate θ. This is done in the next section. However, we do not need it to investigate stability.

The only contribution to the **potential energy** of the plank comes from uniform gravity. This is given by $V = MgZ$, where Z is the vertical displacement of the centre of mass G of the plank. Elementary trigonometry (see Figure 9.6) shows that $Z = a\cos\theta + a\theta\sin\theta - a$, so that

$$V = Mga(\cos\theta + \theta\sin\theta - 1).$$

We must now show that the constraint forces do no work. The rate of working of the constraint force $\boldsymbol{R}$ that the log exerts on the plank is $\boldsymbol{R}\cdot\boldsymbol{v}^C$, where $\boldsymbol{v}^C$ is the velocity of the particle C of the plank that is *instantaneously* in contact with the log. But, since the plank rolls on the log, $\boldsymbol{v}^C = \boldsymbol{0}$ so that the rate of working of $\boldsymbol{R}$ is zero. Also, the internal constraint forces that enforce the rigidity of the plank do no work in total. Hence, the **constraint forces** do no work in total.

Energy conservation therefore applies in the form

$$T + Mga(\cos\theta + \theta\sin\theta - 1) = E.$$

It follows that the equilibrium position (with the plank on top of the log) will be stable if V has a *minimum* at $\theta = 0$. Now $V' = Mga\theta\cos\theta$ and $V'' = Mga(\cos\theta - \theta\sin\theta)$ so that, when $\theta = 0$, $V' = 0$ and $V'' = 1$. Hence V has a minimum at $\theta = 0$ and so the **equilibrium position** is stable. ■

9.4 KINETIC ENERGY OF A RIGID BODY

The general theory we have presented applies to *any* multi-particle system; in particular, it applies to the rigid array of particles that we call a **rigid body**. However, in

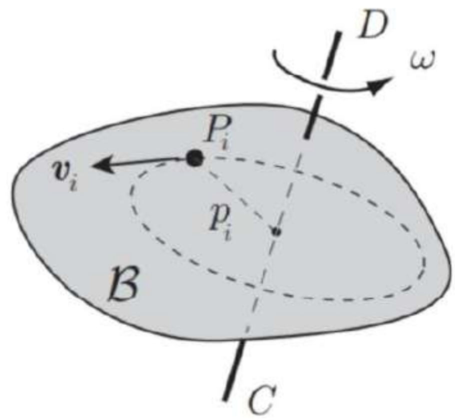

FIGURE 9.7 The rigid body $\mathcal{B}$ rotates about the *fixed* axis CD with angular velocity ω. A typical particle P_i moves on the circular path shown.

order to make use of energy conservation in rigid body dynamics, we need to be able to express the **kinetic energy** T of the body in terms of the generalised coordinates.

Rigid body with a fixed axis

Figure 9.7 shows a rigid body $\mathcal{B}$ which is rotating about the *fixed* axis CD. (Imagine that the body is penetrated by a thin light spindle, which is smoothly pivoted in a fixed position.) A typical particle P_i of the body can move on the circular path shown. This circle has radius p_i, where p_i is the perpendicular distance of P_i from the axis CD. Suppose that, at some instant, the angular velocity of $\mathcal{B}$ about the axis CD is ω. Then the *speed* of particle P_i at this instant is $|\omega| p_i$, and its kinetic energy is $\frac{1}{2}m_i\,(\omega p_i)^2$. The total kinetic energy of $\mathcal{B}$ is therefore

$$T = \sum_{i=1}^{N} \left(\tfrac{1}{2}m_i\,(\omega p_i)^2\right) = \tfrac{1}{2}\left(\sum_{i=1}^{N} m_i\,{p_i}^2\right)\omega^2.$$

Definition 9.2 ***Moment of inertia*** *The quantity*

$$\boxed{I_{CD} = \sum_{i=1}^{N} m_i\,{p_i}^2} \qquad (9.8)$$

where p_i is the perpendicular distance of the mass m_i from the axis CD, is called the ***moment of inertia*** *of the body $\mathcal{B}$ about the axis CD.*

The **moment of inertia**, as defined above, does not depend on the motion of the body $\mathcal{B}$. It is a purely *geometrical* quantity (like centre of mass), which describes how the mass in $\mathcal{B}$ is distributed relative to the axis CD. The further the mass in $\mathcal{B}$ lies from the axis, the larger is the moment of inertia of $\mathcal{B}$ about that axis. In the theory of rotating rigid bodies, the moment of inertia plays a similar rôle to that played by mass in the translational motion of a particle.

Our result may be summarised as follows:

Kinetic energy of a rigid body with a fixed axis

Suppose the rigid body $\mathcal{B}$ is rotating about the fixed axis CD with angular velocity ω. Then the kinetic energy of $\mathcal{B}$ is given by

$$T = \tfrac{1}{2} I_{CD}\, \omega^2, \tag{9.9}$$

where I_{CD} is the moment of inertia of $\mathcal{B}$ about the axis CD.

Example 9.8 *Moment of inertia of a hoop*

Find the moment of inertia of a uniform hoop of mass M and radius a about its axis of rotational symmetry.

Solution

This is the easiest case to treat since each particle of the hoop has perpendicular distance a from the specified axis. The required moment of inertia is therefore

$$I = \sum_{i=1}^{N} m_i\, a^2 = \left(\sum_{i=1}^{N} m_i\right) a^2 = M a^2,$$

where M is the mass of the whole hoop. ■

It is evident that, in order to solve problems that include rotating rigid bodies, we need to know their moments of inertia. These can be worked out from the definition (9.8), or its counterpart for continuous mass distributions. The Appendix at the end of the book contains examples of how to do this and also contains a table of common moments of inertia, including those for the uniform **rod, hoop, disk** and **sphere**. Most readers will find it convenient to remember the moments of inertia in these four cases.

Example 9.9 *Rotational kinetic energy of the Earth*

Estimate the rotational kinetic energy of the Earth, regarded as a rigid uniform sphere rotating about a *fixed* axis through its centre.

Solution

From the Appendix, we find that I, the moment of inertia of a uniform sphere about an axis through its centre is given by $I = 2MR^2/5$, where M is the mass of the sphere and R its radius. The kinetic energy of the Earth is therefore given by

$$T = \tfrac{1}{2} I \omega^2 = \tfrac{1}{5} M R^2 \omega^2,$$

where M is the mass the Earth, R is its radius, and ω is its angular velocity.

On inserting the values $M = 6.0 \times 10^{24}$ kg, $R = 6400$ km and $\omega = 7.3 \times 10^{-5}$ radians per second, $T = 2.6 \times 10^{29}$ J approximately. ■

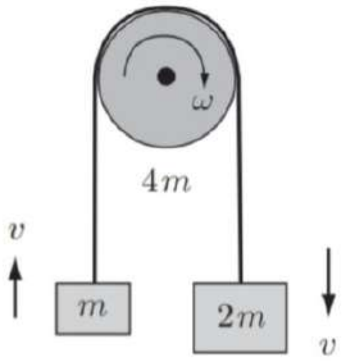

FIGURE 9.8 Two blocks of masses m and $2m$ are connected by a light inextensible string which passes over a circular pulley of mass $4m$ and radius a.

Example 9.10 *Attwood's machine*

Two blocks of masses m and $2m$ are connected by a light inextensible string which passes over a uniform circular pulley of radius a and mass $4m$. Find the upward acceleration of the mass m.

Solution

The system is shown in Figure 9.8. We suppose that the string does not slip on the pulley and that the pulley is smoothly pivoted about its axis of symmetry.

Let z be the upward displacement of the mass m (from some reference configuration) and v $(= \dot{z})$ its upward velocity at time t. Then, since the string is *inextensible*, the mass $2m$ must have the same displacement and velocity, but measured downwards. The angular velocity ω of the pulley is determined from the condition that the string does not slip. In this case, the velocity of the rim of the pulley and the velocity of the string must be the same at each point where they are in contact, that is, $a\omega = v$. Hence $\omega = v/a$. Also, from the table in the Appendix, the moment of inertia of a uniform circular disk of mass M and radius a about its axis of symmetry is $\frac{1}{2}Ma^2$. Hence, the total **kinetic energy** of the system is

$$T = \tfrac{1}{2}mv^2 + \tfrac{1}{2}(2m)v^2 + \tfrac{1}{2}\left(\tfrac{1}{2}(4m)a^2\right)\left(\frac{v}{a}\right)^2 = \tfrac{5}{2}mv^2.$$

The gravitational **potential energy** of the system (relative to the reference configuration) is

$$V = mgz - (2m)gz = -mgz.$$

We must now dispose of the **constraint forces.** (i) At the smooth pivot that supports the pulley, the reactions are perpendicular to the velocities of the particles on which they act. Hence these reactions do no work. (ii) Since there is no slippage between the string and the three material bodies of the system, the total work done by the string on the bodies must be equal and opposite to the total work done by the bodies on the string.* (iii) The internal forces that keep the pulley rigid do no work in total. Hence the constraint forces do no work in total.

* Since this string is massless and inextensible, it can have neither kinetic nor potential energy so that the total work done on the string must actually be zero.

Energy conservation therefore applies in the form

$$\tfrac{5}{2}mv^2 - mgz = E,$$

where E is the total energy. If we now differentiate this equation with respect to t (and cancel by mv), we obtain

$$\frac{dv}{dt} = \tfrac{1}{5}g$$

which is the **equation of motion** of the system. Thus the upward **acceleration** of the mass m is $g/5$. (If the pulley were massless, the result would be $g/3$.) ■

Rigid body in general motion

We now go on to find the kinetic energy of a rigid body that has translational as well as rotational motion. The method depends on the following theorem.

Theorem 9.1 *Suppose a general system of particles $\mathcal{S}$ has total mass M and that its centre of mass G has velocity $\boldsymbol{V}$. Then the total kinetic energy of $\mathcal{S}$ can be written in the form*

$$T = \tfrac{1}{2}MV^2 + T^G, \tag{9.10}$$

*where $V = |\boldsymbol{V}|$ and T^G is the kinetic energy of $\mathcal{S}$ in its motion **relative to** G.*

Proof. By definition,

$$\begin{aligned}
T^G &= \frac{1}{2}\sum_{i=1}^{N} \tfrac{1}{2}m_i\,(\boldsymbol{v}_i - \boldsymbol{V})\cdot(\boldsymbol{v}_i - \boldsymbol{V})\\
&= \frac{1}{2}\sum_{i=1}^{N} m_i\boldsymbol{v}_i\cdot\boldsymbol{v}_i - \frac{1}{2}\left(\sum_{i=1}^{N} m_i\boldsymbol{v}_i\right)\cdot\boldsymbol{V} - \frac{1}{2}\boldsymbol{V}\cdot\left(\sum_{i=1}^{N} m_i\boldsymbol{v}_i\right) + \frac{1}{2}\left(\sum_{i=1}^{N} m_i\right)\boldsymbol{V}\cdot\boldsymbol{V}\\
&= T - \tfrac{1}{2}(M\boldsymbol{V})\cdot\boldsymbol{V} - \tfrac{1}{2}\boldsymbol{V}\cdot(M\boldsymbol{V}) + \tfrac{1}{2}M(\boldsymbol{V}\cdot\boldsymbol{V})\\
&= T - \tfrac{1}{2}MV^2,
\end{aligned}$$

as required. ■

The term $\frac{1}{2}MV^2$ can be regarded as the **translational** contribution to T. When the system $\mathcal{S}$ is a **rigid body**, T^G also has a nice physical interpretation. In this case, the motion of $\mathcal{S}$ relative to G is an angular velocity ω about an axis CD passing through G, as shown in Figure 9.9. It then follows from equation (9.9) that $T^G = \frac{1}{2}I_{CD}\,\omega^2$. This can be regarded as the **rotational** contribution to T. We therefore have the result:

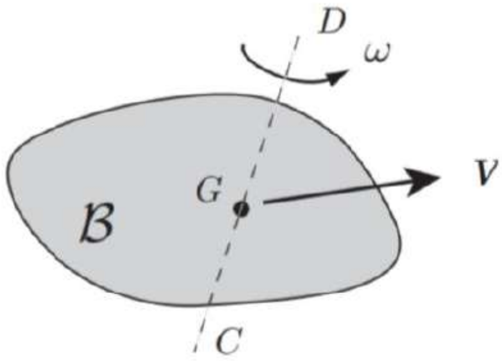

FIGURE 9.9 A rigid body $\mathcal{B}$ in general motion. The centre of mass G has velocity V and $\mathcal{B}$ is also rotating with angular velocity ω about an axis through G.

Kinetic energy of a rigid body in general motion

Let $\mathcal{B}$ be a rigid body of mass M and and let G be its centre of mass. Suppose that G has velocity $\boldsymbol{V}$ and that the body is also rotating with angular velocity ω about an axis CD passing through G. Then the **kinetic energy** of $\mathcal{B}$ is given by

$$T = \tfrac{1}{2}MV^2 + \tfrac{1}{2}I_{CD}\,\omega^2, \tag{9.11}$$

where $V = |\boldsymbol{V}|$ and I_{CD} is the moment of inertia of $\mathcal{B}$ about the axis CD. The term $\frac{1}{2}MV^2$ is called the **translational** kinetic energy and the term $\frac{1}{2}I_{CD}\,\omega^2$ the **rotational** kinetic energy of $\mathcal{B}$.

Example 9.11 *Kinetic energy of a rolling wheel*

Find the kinetic energy of the rolling wheel shown in Figure 2.8.

Solution

Assume the wheel to be uniform with mass M and radius b. Then its centre of mass C has speed u so that the **translational** kinetic energy is $\frac{1}{2}Mu^2$. Because of the rolling condition, the angular velocity of the wheel is given by $\omega = u/b$ so that the **rotational** kinetic energy is $\frac{1}{2}I(u/b)^2$, where $I = \frac{1}{2}Mb^2$. The **total** kinetic energy of the wheel is therefore given by

$$T = \tfrac{1}{2}Mu^2 + \tfrac{1}{2}\left(\tfrac{1}{2}Mb^2\right)\left(\frac{u}{b}\right)^2 = \frac{3Mu^2}{4}. \blacksquare$$

Example 9.12 *Cylinder rolling down a plane*

A uniform hollow circular cylinder is rolling down a *rough* plane inclined at an angle α to the horizontal. Find the acceleration of the cylinder.

Solution

Suppose that, at time t, the cylinder has displacement x down the plane (from some reference configuration) and that the centre of mass G of the cylinder has velocity

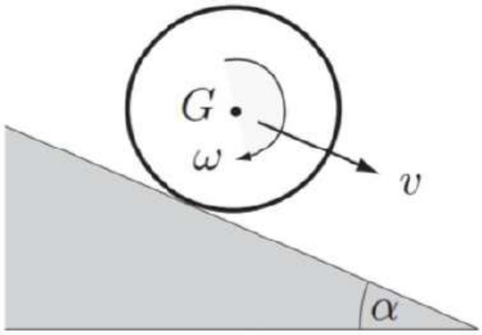

FIGURE 9.10 A hollow circular cylinder rolls down a plane inclined at angle α to the horizontal.

$v\ (=\dot{x})$ down the plane. The angular velocity ω of the cylinder is then determined by the rolling condition to be $\omega = v/b$. The kinetic energy of the cylinder is therefore

$$T = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\left(\frac{v}{b}\right)^2$$

where M is the mass of the cylinder, and I is its moment of inertia about its axis of symmetry. From the Appendix, we find that $I = Mb^2$ so that the **kinetic energy** of the cylinder is given by $T = Mv^2$.

The gravitational **potential energy** of the cylinder is given by $V = -Mgx\sin\alpha$.

We must now dispose of the constraint forces. The reaction forces that the inclined plane exerts on the cylinder act on particles of the cylinder which, because of the rolling condition, have zero velocity. These reaction forces therefore do no work. Also the internal forces that keep the cylinder rigid do no work in total. Hence the **constraint forces** do no work in total.

Conservation of energy therefore applies in the form

$$Mv^2 - Mgx\sin\alpha = E,$$

where E is the total energy. If we now differentiate this equation with respect to t (and cancel by Mv), we obtain

$$\frac{dv}{dt} = \tfrac{1}{2}g\sin\alpha,$$

which is the **equation of motion** of the cylinder. Thus the **acceleration** of the cylinder down the plane is $\frac{1}{2}g\sin\alpha$. (A block sliding down a *smooth* plane would have acceleration $g\sin\alpha$.) ■

Example 9.13 *The sliding ladder*

A uniform ladder of length $2a$ is supported by a smooth horizontal floor and leans against a smooth vertical wall.* The ladder is released from rest in a position making an angle of 60° with the downward vertical. Find the energy conservation equation for the ladder.

* Don't try this at home!

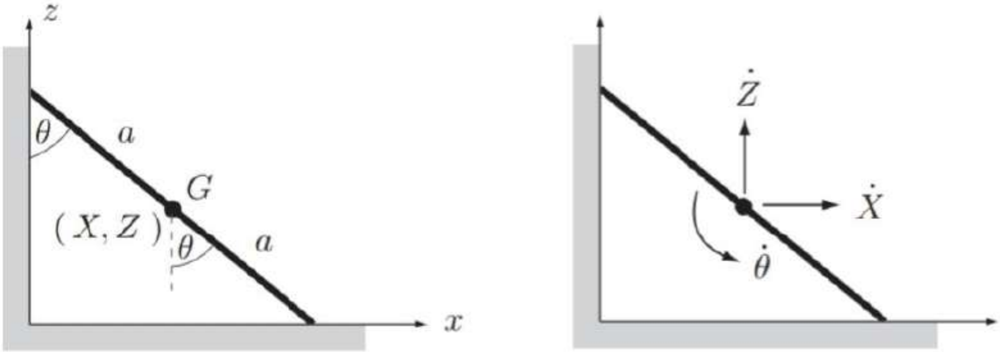

FIGURE 9.11 A uniform ladder of mass M and length $2a$ is supported by a smooth horizontal floor and leans against a smooth vertical wall. At time t, its centre of mass G has (x, z)-coordinates (X, Z) and the ladder makes an angle θ with the downward vertical.

Solution

Let θ be the angle that the ladder makes with the downward vertical after time t. The (x, z)-coordinates of the centre of mass G are then given by

$$X = a\sin\theta, \qquad Z = a\cos\theta,$$

and the corresponding velocity components by

$$\dot{X} = (a\cos\theta)\dot{\theta}, \qquad \dot{Z} = -(a\sin\theta)\dot{\theta}.$$

The angular velocity of the ladder at time t is simply $\dot{\theta}$ (see Figure 9.11). The **kinetic energy** of the ladder is therefore given by

$$T = \tfrac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2\right) + \tfrac{1}{2}I\dot{\theta}^2 = \tfrac{1}{2}Ma^2\dot{\theta}^2 + \tfrac{1}{2}I\dot{\theta}^2,$$

where M is the mass of the ladder and I is its moment of inertia about the horizontal axis through G. From the Appendix, we find that $I = Ma^2/3$ so that the **kinetic energy** of the ladder is given by $T = (2Ma^2/3)\dot{\theta}^2$.

The gravitational **potential energy** of the ladder is given by $V = MgZ = Mga\cos\theta$.

We must now dispose of the constraint forces. The reaction forces that the smooth floor and wall exert on the ladder are both perpendicular to the particles of the ladder on which they act. These reaction forces therefore do no work. Also, the internal forces that keep the ladder rigid do no work in total. Hence the **constraint forces** do no work in total.

Conservation of energy therefore applies in the form

$$\tfrac{2}{3}Ma^2\dot{\theta}^2 + Mga\cos\theta = E,$$

where E is the total energy. From the initial conditions $\dot{\theta} = 0$ and $\theta = \pi/3$ when $t = 0$, it follows that $E = \tfrac{1}{2}Mga$. The **energy conservation equation** for the ladder

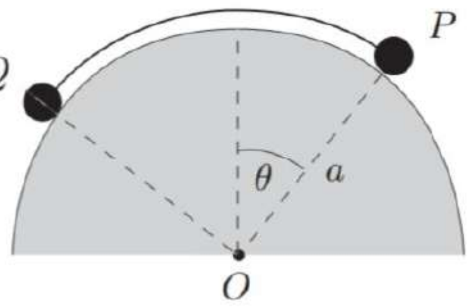

FIGURE 9.12 Two particles P and Q are connected by a light inextensible string and can move, with the string taut, on the surface of a smooth horizontal cylinder.

is therefore

$$\dot{\theta}^2 = \frac{3g}{4a}(1 - 2\cos\theta).$$

Since the system has only one degree of freedom, this equation is sufficient to determine the motion.

A curious feature of this problem (not proved here) is that the ladder does not maintain contact with the wall all the way down, but leaves the wall when θ becomes equal to $\cos^{-1}(1/3) \approx 71°$. ■

Problems on Chapter 9

Answers and comments are at the end of the book.
Harder problems carry a star (∗).

Potential energy and stability

9.1 Figure 9.12 shows two particles P and Q, of masses M and m, that can move on the smooth outer surface of a fixed horizontal cylinder. The particles are connected by a light inextensible string of length $\pi a/2$. Find the equilibrium configuration and show that it is unstable.

9.2 A uniform rod of length $2a$ has one end smoothly pivoted at a fixed point O. The other end is connected to a fixed point A, which is a distance $2a$ vertically above O, by a light elastic spring of natural length a and modulus $\frac{1}{2}mg$. The rod moves in a vertical plane through O. Show that there are two equilibrium positions for the rod, and determine their stability. [The vertically upwards position for the rod would compress the spring to zero length and is excluded.]

9.3 The internal potential energy function for a diatomic molecule is approximated by the **Morse potential**

$$V(r) = V_0\left(1 - e^{-(r-a)/b}\right)^2 - V_0,$$

where r is the distance of separation of the two atoms, and V_0, a, b are positive constants. Make a sketch of the Morse potential.

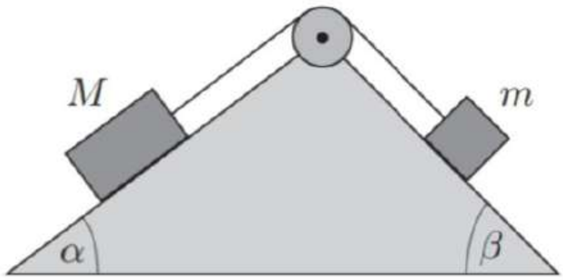

FIGURE 9.13 Two blocks of masses M and m slide on smooth planes inclined at angles α and β to the horizontal. The blocks are connected by a light inextensible string that passes over a light frictionless pulley.

Suppose the molecule is restricted to *vibrational* motion in which the centre of mass G of the molecule is fixed, and the atoms move on a fixed straight line through G. Show that there is a single equilibrium configuration for the molecule and that it is stable. If the atoms each have mass m, find the angular frequency of small vibrational oscillations of the molecule.

9.4* The internal gravitational potential energy of a system of masses is sometimes called the **self energy** of the system. (The reference configuration is taken to be one in which the particles are all a great distance from each other.) Show that the self energy of a uniform sphere of mass M and radius R is $-3M^2G/5R$. [Imagine that the sphere is built up by the addition of successive thin layers of matter brought in from infinity.]

Particles only

9.5 Figure 9.13 shows two blocks of masses M and m that slide on smooth planes inclined at angles α and β to the horizontal. The blocks are connected by a light inextensible string that passes over a light frictionless pulley. Find the acceleration of the block of mass m up the plane, and deduce the tension in the string.

9.6 Consider the system shown in Figure 9.12 for the special case in which the particles P, Q have masses $2m$, m respectively. The system is released from rest in a symmetrical position with θ, the angle between OP and the upward vertical, equal to $\pi/4$. Find the energy conservation equation for the subsequent motion in terms of the coordinate θ.

* Find the normal reactions of the cylinder on each of the particles. Show that P is first to leave the cylinder and that this happens when $\theta = 70^\circ$ approximately.

Ropes

9.7 A heavy uniform rope of length $2a$ is draped symmetrically over a *thin* smooth horizontal peg. The rope is then disturbed slightly and begins to slide off the peg. Find the speed of the rope when it finally leaves the peg.

9.8 A uniform heavy rope of length a is held at rest with its two ends close together and the rope hanging symmetrically below. (In this position, the rope has two long vertical segments connected by a small curved segment at the bottom.) One of the ends is then released. Find the velocity of the free end when it has descended by a distance x.

Deduce a similar formula for the acceleration of the free end and show that it always *exceeds* g. Find how far the free end has fallen when its acceleration has risen to $5g$.

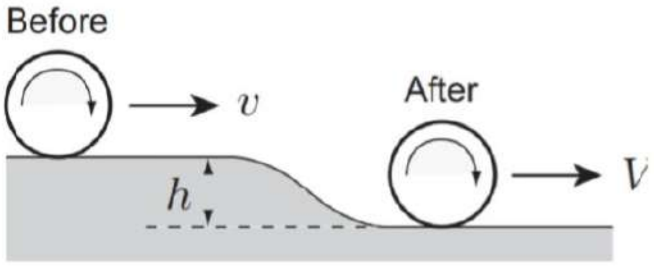

FIGURE 9.14 The circular hoop *rolls* down the slope from one level to another.

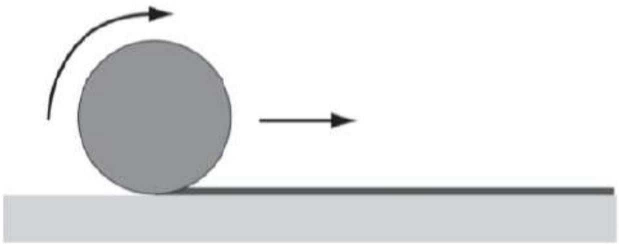

FIGURE 9.15 The roll of paper moves to the right and the free paper is gathered on to the roll.

9.9 A heavy uniform rope of mass M and length $4a$ has one end connected to a fixed point on a smooth horizontal table by light elastic spring of natural length a and modulus $\frac{1}{2}Mg$, while the other end hangs down over the edge of the table. When the spring has its natural length, the free end of the rope hangs a distance a vertically below the level of the table top. The system is released from rest in this position. Show that the free end of the rope executes simple harmonic motion, and find its period and amplitude.

Rigid bodies

9.10 A circular hoop is rolling with speed v along level ground when it encounters a slope leading to more level ground, as shown in Figure 9.14. If the hoop loses altitude h in the process, find its final speed.

9.11 A uniform ball is rolling in a straight line down a *rough* plane inclined at an angle α to the horizontal. Assuming the ball to be in planar motion, find the energy conservation equation for the ball. Deduce the acceleration of the ball.

9.12 A uniform circular cylinder (a yo-yo) has a light inextensible string wrapped around it so that it does not slip. The free end of the string is secured to a fixed point and the yo-yo descends in a vertical straight line with the straight part of the string also vertical. Explain why the string does no work on the yo-yo. Find the energy conservation equation for the yo-yo and deduce its acceleration.

9.13 Figure 9.15 shows a partially unrolled roll of paper on a horizontal floor. Initially the paper on the roll has radius a and the free paper is laid out in a straight line on the floor. The roll is then projected horizontally with speed V in such a way that the free paper is gathered up on to the roll. Find the speed of the roll when its radius has increased to b. [Neglect the bending stiffness of the paper.] Deduce that the radius of the roll when it comes to rest is

$$a\left(\frac{3V^2}{4ga}+1\right)^{1/3}.$$

9.14 A rigid body of general shape has mass M and can rotate freely about a fixed horizontal axis. The centre of mass of the body is distance h from the rotation axis, and the moment of inertia of the body about the rotation axis is I. Show that the period of small oscillations of the body about the downward equilibrium position is

$$2\pi\left(\frac{I}{Mgh}\right)^{1/2}.$$

Deduce the period of small oscillations of a uniform rod of length $2a$, pivoted about a horizontal axis perpendicular to the rod and distance b from its centre.

9.15 A uniform ball of radius a can roll without slipping on the *outside* surface of a fixed sphere of (outer) radius b and centre O. Initially the ball is at rest at the highest point of the sphere when it is slightly disturbed. Find the speed of the centre G of the ball in terms of the variable θ, the angle between the line OG and the upward vertical. [Assume planar motion.]

9.16 A uniform ball of radius a and centre G can roll without slipping on the *inside* surface of a fixed hollow sphere of (inner) radius b and centre O. The ball undergoes planar motion in a vertical plane through O. Find the energy conservation equation for the ball in terms of the variable θ, the angle between the line OG and the downward vertical. Deduce the period of small oscillations of the ball about the equilibrium position.

9.17* Figure 9.6 shows a uniform thin rigid plank of length $2b$ which can roll without slipping on top of a rough circular log of radius a. The plank is initially in equilibrium, resting symmetrically on top of the log, when it is slightly disturbed. Find the period of small oscillations of the plank.