VIVEKANANDA COLLEGE
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NAAC ACCREDITED ‘A’ GRADE

Topic: Caylay-Hamilton Theorem

Course Title:

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Matric polynomials

We know that a function $f_n(x)$ of the form

$$f_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \quad (a_0 \neq 0)$$

is said to be a polynomial of degree n with real coefficients $(n \in \mathbb{N})$

Now an expression of the form

$$P(x) = A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n,$$

where $A_0, A_1, A_2, \ldots, A_n$ are all square matrices of same order, is called a matric polynomial of degree n. (Here similarly A_0 is not a null matrix). $(n \in \mathbb{N})$

Let us conciders a ~~two~~ 2×2 matrix $A = \begin{pmatrix} x^3+2x+5 & x^2+9 \\ 3x+5 & 9x^3+5x^2 \end{pmatrix}$

Here the elements of A are polynomials of x ~~whose~~ and the highest degree of these polynomials is 3.

So, A can be expressed as a matric polynomial of degree 3 as

$$A = \begin{pmatrix} x^3+2x+5 & x^2+9 \\ 3x+5 & 9x^3+5x^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}x^3 + \begin{pmatrix} 0 & 1 \\ 0 & 5 \end{pmatrix}x^2 + \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}x + \begin{pmatrix} 5 & 9 \\ 5 & 0 \end{pmatrix}$$

In general a $m\times m$ real matrix A whose elements are real polynomial in x can be expressed as a matric polynomial whose co-efficients are $m\times m$ real matrix.

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Sum and product of two Matric Polynomials

* Two matric polynomials in x are said to be equal if they are of same degree and the co-efficients of the ~~same~~ equal powers of x are the same matrices.

* Let $P(x)$ and $Q(x)$ be two matric polynomials, whose co-efficients are square matrices of same order. Then the sum $P(x)+Q(x)$ and the product $P(x)Q(x)$ are defined as below:

If $P(x) = A_0 x^n + A_1 x^{n-1} + \cdots + A_{n-1} x + A_n$

and $Q(x) = B_0 x^m + B_1 x^{m-1} + \cdots + B_{m-1} x + B_m$

if $n > m$ then

$$P(x)+Q(x) = A_0 x^n + A_1 x^{n-1} + \cdots + A_{n-m-1} x^{m+1} + (A_{n-m}+B_0) x^m$$
$$+ (A_{n-m+1} + B_1) x^{m-1} + \cdots + (A_{n-1} + B_{m-1}) x$$
$$+ (A_n + B_m)$$

So, clearly $P(x)+Q(x) = Q(x)+P(x)$

But matrix multiplication is non-commutative, so in general $P(x)Q(x) \neq Q(x)P(x)$

* You can check it by taking

$$P(x) = \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix} x^2 + \begin{pmatrix} 2 & 6 \\ 4 & 1 \end{pmatrix} x + \begin{pmatrix} -1 & 3 \\ 0 & 5 \end{pmatrix} \text{ and}$$

$$Q(x) = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} x + \begin{pmatrix} -1 & 0 \\ 3 & 1 \end{pmatrix}$$

You will get $P(x)+Q(x) = Q(x)+P(x)$

and ~~that~~ $P(x)Q(x) \neq Q(x)P(x)$

* Characteristic Equation :-

We take a $n \times n$ matrix A over a field F, that is all elements of A are taken from the field F. Then the matric polynomial $xI_n - A$ of degree 1 is called the characteristic matrix of A, where I_n is the unit matrix of order n.

* The determinant of $(A - xI_n)$ is called the characteristic polynomial of A which is a ordinarly polynomial of degree n and it is denoted by $\phi_A(x)$.

The equation $\phi_A(x) = 0$ is called the characteristic equation of A.

i.e. if $A = (a_{ij})_{n\times n}$ then the characteristic equation of A is

$$\phi_A(x) = \begin{vmatrix} a_{11}-x & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-x & \cdots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-x \end{vmatrix} = 0$$

Example:— $A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$

Then the characteristic equation of A is $|A - xI_3| = 0$

i.e. $\begin{vmatrix} 0-x & 0 & 1 \\ 3 & 1-x & 0 \\ -2 & 1 & 4-x \end{vmatrix} = 0$

or, $(-x)\{(1-x)(4-x) - 0\} + 1\{3 + 2(1-x)\} = 0$

or, $(-x)(x^2 - 5x + 4) + (5 - 2x) = 0$

or, $-x^3 + 5x^2 - 4x + 5 - 2x = 0$

or, $x^3 - 5x^2 + 6x - 5 = 0$ ——— (i)

∴ the equation i, of degree 3, is the characteristic equation of the 3×3 matrix A.

* Cayley – Hamilton Theorem

Statement :– Every square matrix satisfies its own characteristic equation.

// Proof is out of syllabus //

Illustration :– Let us take the square matrix of the previous example. i.e. $A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$

We have calculated the characteristic equation of A ~~before~~ in the previous example. The characteristic equation of A is $x^3 - 5x^2 + 6x - 5 = 0$ ..... i)

Now Cayley - Hamilton's theorem says that if we replace x by the matrix A itself then it also satisfies the equation. Lets check that

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{pmatrix}$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 5 & 14 \\ -3 & 4 & 15 \\ -13 & 19 & 51 \end{pmatrix}$$

$$-5A^2 = \begin{pmatrix} 10 & -5 & -20 \\ -15 & -5 & -15 \\ 25 & -25 & -70 \end{pmatrix}$$

$$6A = \begin{pmatrix} 0 & 0 & 6 \\ 18 & 6 & 0 \\ -12 & 6 & 24 \end{pmatrix}$$

$A^3 - 5A^2 + 6A - 5I_{3\times3}$ [We replace x^0 by $A^0 = I_{3\times3}$ as A is a 3x3 square matrix]

$$= \begin{pmatrix} -5 & 5 & 14 \\ -3 & 4 & 15 \\ -13 & 19 & 51 \end{pmatrix} + \begin{pmatrix} 10 & -5 & -20 \\ -15 & -5 & -15 \\ 25 & -25 & -70 \end{pmatrix}$$ [$I_{3\times3}$ is 3x3 identity matrix]

$$+ \begin{pmatrix} 0 & 0 & 6 \\ 18 & 6 & 0 \\ -12 & 6 & 24 \end{pmatrix} + \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & -6 \\ -18 & -1 & 0 \\ 12 & -6 & -19 \end{pmatrix} + \begin{pmatrix} -5 & 0 & 6 \\ 18 & 1 & 0 \\ -12 & 6 & 19 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O_{3\times3}$$ [where $O_{3\times3}$ is 3x3 null matrix]

$\therefore \quad A^3 - 5A^2 + 6A - 5I_{3\times3} = O_{3\times3}$

$\therefore$ A satisfies its own characteristic equation given by equation no. 1).

* Note that if the characteristic equation of a square matrix $A_{n\times n}$ be $c_0 x^n + c_1 x^{n-1} + \dots + c_{n-1}x + c_n = 0$ then by Cayley-Hamilton theorem we can write

$$c_0 A^n + c_1 A^{n-1} + \dots + c_{n-1} A + c_n I_{n\times n} = O_{n\times n}$$

where $c_0 = (-1)^n$ ~~[illegible]~~

and $c_r = (-1)^{n-r}$ [Sum of the principal minors of A of order r]

in particular $c_n = \det A$

* Applications of Cayley-Hamilton theorem

□ We can use Cayley-Hamilton theorem to compute A^{-1}.

Let the characteristic equation of $A_{n\times n}$ be

$c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n = 0$, so, by

Cayley-Hamilton theorem we can write

$$c_0 A^n + c_1 A^{n-1} + \cdots + c_{n-1} A + c_n I_{n\times n} = O_{n\times n} \longrightarrow (i)$$

Now as $c_n = \det A$, if $|A| \neq 0$ i.e. $\det A \neq 0$, $c_n \neq 0$

$\therefore c_n^{-1}$ exists and A^{-1} exists.

Multiplying i) by c_n^{-1} we have

$$c_n^{-1}\{c_0 A^n + c_1 A^{n-1} + \cdots + c_{n-1} A\} + I_{n\times n} = O_{n\times n}$$

or, $(-c_n^{-1})\{c_0 A^n + c_1 A^{n-1} + \cdots + c_{n-1} A\} = I_{n\times n}$

or, $A\{(-c_n^{-1})(c_0 A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1})\} = I_{n\times n}$

As A^{-1} exists, multiplying the last equation by A^{-1} we will get

$$(-c_n^{-1}).(c_0 A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1}) = A^{-1} \longrightarrow (ii)$$

$\therefore A^{-1}$ can be calculated by the equation (ii) whose co-efficients are ~~scal~~ scalars.

Illustration :- We take a non-singular matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix}$

where $|A| = 4$

The characteristic equation of A is

$$\begin{vmatrix} 1-x & 0 & 0 \\ 1 & 2-x & 1 \\ 2 & 0 & 2-x \end{vmatrix} = 0$$

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or, $(1-x)(2-x)(2-x)=0$

or, $(1-x)(4-4x+x^2)=0$

or, $4-4x+x^2-4x+4x^2-x^3=0$

or, $-x^3+5x^2-8x+4=0$

or, $x^3-5x^2+8x-4=0$

By Cayley-Hamilton theorem, we get

$$A^3-5A^2+8A-4I_{3\times3}=O_{3\times3}$$

or, $4I_{3\times3}=A^3-5A^2+8A$

or, $4I_{3\times3}=A(A^2-5A+8I)$

or, $I_3=A\{\frac{1}{4}(A^2-5A+8I)\}$

or, $A^{-1}=\frac{1}{4}(A^2-5A+8I)$

Now $A^2=\begin{pmatrix}1&0&0\\1&2&1\\2&0&2\end{pmatrix}\begin{pmatrix}1&0&0\\1&2&1\\2&0&2\end{pmatrix}=\begin{pmatrix}1&0&0\\5&4&4\\6&0&4\end{pmatrix}$

$\therefore A^{-1}=\frac{1}{4}\left\{\begin{pmatrix}1&0&0\\5&4&4\\6&0&4\end{pmatrix}-5\begin{pmatrix}1&0&0\\1&2&1\\2&0&2\end{pmatrix}+\begin{pmatrix}8&0&0\\0&8&0\\0&0&8\end{pmatrix}\right\}$

$=\frac{1}{4}\left\{\begin{pmatrix}1&0&0\\5&4&4\\6&0&4\end{pmatrix}+\begin{pmatrix}-5&0&0\\-5&-10&-5\\-10&0&-10\end{pmatrix}+\begin{pmatrix}8&0&0\\0&8&0\\0&0&8\end{pmatrix}\right\}$

$=\frac{1}{4}\begin{pmatrix}4&0&0\\0&2&-1\\-4&0&2\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1/2&-1/4\\-1&0&1/2\end{pmatrix}$

This gives us the required A^{-1} by Cayley-Hamilton theorem.

☐ Another application of Cayley-Hamilton theorem is that we can express A^m $(m \geq n)$ in terms of lower powers of A, where A is a $n \times n$ square matrix.

If $c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n = 0$

then by Cayley-Hamilton's theorem we get

$$c_0 A^n + c_1 A^{n-1} + \cdots + c_{n-1} A + c_n I_{n\times n} = O_{n\times n}$$

or, $-c_0 A^n = c_1 A^{n-1} + c_2 A^{n-2} + \cdots + c_{n-1} A + c_n I_{n\times n}$

[Since $c_0 = (-1)^n$, $c_0 \neq 0$]

or, $A^n = (-c_0^{-1})(c_1 A^{n-1} + c_2 A^{n-2} + \cdots + c_{n-1} A + c_n I_{n\times n})$

Multiplying by A^{m-n} [As $m \geq n$, $m-n \geq 0$] we get

$$A^m = (-c_0^{-1})(c_1 A^{m-1} + c_2 A^{m-2} + \cdots + c_{n-1} A^{m-n+1} + c_n A^{m-n})$$

$\therefore$ A^m is expressible as an expression involving lower powers of A.

<u>Illustration</u>:- As in the last illustration, the characteristic equation of A is $x^3 - 5x^2 + 8x - 4 = 0$.

By Cayley-Hamilton's theorem we get

$A^3 - 5A^2 + 8A - 4I = 0$ [where I is 3×3 identity matrix]

or, $A^3 = 5A^2 - 8A + 4I$

then $A^4 = A \cdot A^3 = A(5A^2 - 8A + 4I) = 5A^3 - 8A^2 + 4A$

similarly $A^5 = A \cdot A^4 = 5A^4 + 8A^3 + 4A^2$

and so on

So, A^m can be expressed as an expression involving lower powers (less than m) of A where $m \geq 3$.