VIVEKANANDA COLLEGE, THAKURPUKUR

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NAAC ACCREDITED ‘A’ GRADE

Topic :	Multivariate Calculus-II
Course Title :	Green’s theorem
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Unit :	Unit-2
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Name of the Department:	Mathematics

(13) $f_z(x,y,z) = 3ye^{3z}$

Integrating (11) with respect to x, we get

(14) $f(x,y,z) = xy^2 + g(y,z)$

where $g(y,z)$ is a constant with respect to x. Then differentiating (14) with respect to y, we have

$$f_y(x,y,z) = 2xy + g_y(y,z)$$

and comparision with (12) gives

$$g_y(y,z) = e^{3z}$$

Thus, $g(y,z) = ye^{3z} + h(z)$ and we rewrite (14) as

$$f(x,y,z) = xy^2 + ye^{3z} + h(z)$$

Finally, differentiating with respect to z and comparing with (13), we obtain $h'(z) = 0$ and therefore, $h(z) = K$, a constant. The desired function is

$$f(x,y,z) = xy^2 + ye^{3z} + K.$$

It is easily verified that $\nabla f = F$

Green's Theorem

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C. (See Fig. 1. We assume that D consists of all points inside C as well as all points on C). In stating Green's Theorem we use the convention that the positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C. Thus, if C is given by the vector function $r(t)$, $a \le t \le b$, then the region D is always on the left as the point $r(t)$ traverses C. (See Fig 2.)

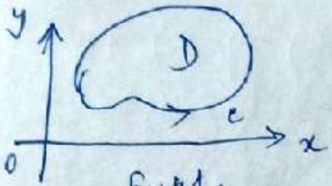


Fig1.

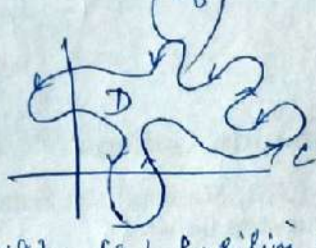


Fig2. (a) Positive orientation (b) Negative orientation

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Note The notation $\oint_C P\,dx + Q\,dy$ or $\oint^{\circlearrowleft}_C P\,dx + Q\,dy$ is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C. Another notation for the positively oriented boundary curve of D is ∂D, so the equation of Green's theorem can be written as

$$(1) \qquad \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$$

Proof. Proof of Green's theorem for the case in which D is a simple region. Notice that Green's Theorem will be proved if we can show that

$$(2) \qquad \int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$$

and

$$(3) \qquad \int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$$

We prove Eq^n (2) by expressing D as a type I region:

$$D = \left\{(x,y) \;:\; a \le x \le b,\; g_1(x) \le y \le g_2(x)\right\}$$

where g_1 and g_2 are continuous functions. This enables us to compute the double integral on the right side of Eqn 2 as follows:

$$(4) \quad \iint_D \frac{\partial P}{\partial y}\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x,y)\, dy\, dx = \int_a^b \left[P(x, g_2(x)) - P(x, g_1(x))\right] dx$$

where the last step follows from the Fundamental Theorem of Calculus.

Now we compute the left side of Eqn. 2 by breaking up C as the union of the four curves C_1, C_2, C_3 and C_4 shown in Fig 3. On C_1 we take x as the parameter and write the parametric equations as $x = x$, $y = g_1(x)$, $a \le x \le b$. Thus

$$\int_{C_1} P(x,y)\,dx = \int_a^b P(x, g_1(x))\,dx$$

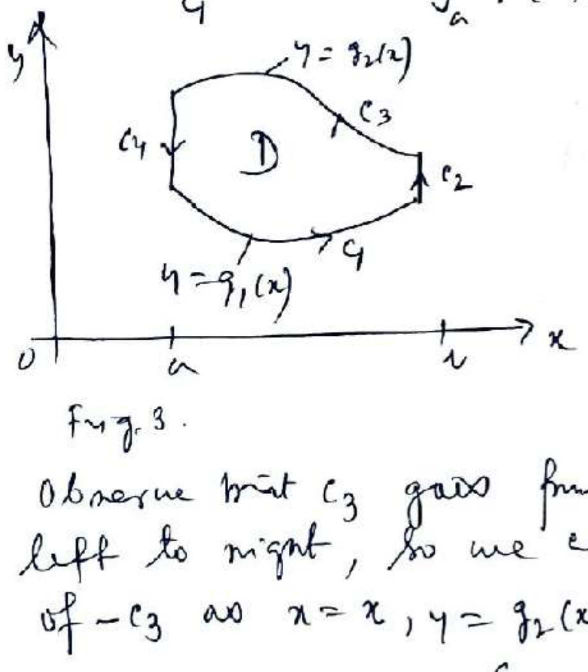


Fig. 3.

Observe that C_3 goes from right to left but $-C_3$ goes from left to right, so we can write the parametric equations of $-C_3$ as $x = x$, $y = g_2(x)$, $a \le x \le b$. Therefore

$$\int_{C_3} P(x,y)\,dx = -\int_{-C_3} P(x,y)\,dx = -\int_a^b P(x, g_2(x))\,dx$$

On C_2 or C_4 (either of which might reduce to just a single point), x is constant, so $dx = 0$ and

$$\int_{C_2} P(x,y)\,dx = 0 = \int_{C_4} P(x,y)\,dx$$

Hence $\int_C P(x,y)\,dx = \int_{C_1} P(x,y)\,dx + \int_{C_2} P(x,y)\,dx$
$+ \int_{C_3} P(x,y)\,dx + \int_{C_4} P(x,y)\,dx$

$$= \int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx$$

Comparing this expression with the one in Equation 4, we see that
$$\int_C P(x,y)\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$$
Eqn. 3 can be proved in much the same way by expressing D as a type II region. Then, by adding Eqns 2 and 3, we obtain Green's theorem.

<u>Example 1</u> Evaluate $\int_C x^4 dx + xy\,dy$, where C is the triangular curve consisting of the line segments from (0, 0) to (1, 0), from (1, 0) to (0, 1) and from (0, 1) to (0, 0) to evaluate the integral.

Soln: We now use Green's Theorem to evaluate the integral. Notice that the region D enclosed by C is simple and C has positive orientation (see Fig 4). If we let $P(x,y) = x^4$ and $Q(x,y) = xy$, then we have

$$\int_C x^4 dx + xy\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^{1-x} (y-0)\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{2}y^2\right]_{y=0}^{y=1-x} dx = \frac{1}{2}\int_0^1 (1-x)^2 dx$$

$$= \frac{1}{6}$$

Example 2 Evaluate $\oint (3y - e^{\sin x})dx + (7x + \sqrt{y^4+1})dy$

where C is the circle $x^2+y^2 = 9$

Soln: The region D bounded by C is the disk $x^2+y^2 \le 9$, So let's change to polar coordinates after applying Green's Theorem:

$$\oint_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4+1})dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}(7x + \sqrt{y^4+1}) - \frac{\partial}{\partial y}(3y - e^{\sin x}) \right] dA$$

$$= \int_0^{2\pi}\int_0^3 (7-3)\, r\,dr\,d\theta$$

$$= 4\int_0^{2\pi} d\theta \int_0^3 r\,dr = 36\pi$$

In Examples 1 and 2 we found that the double integral was easier to evaluate than the line integral. But sometimes it is easier to evaluate the line integral, and Green's Theorem is used in the reverse direction. For instance, if it is known that $P(x,y) = Q(x,y) = 0$ on the curve C, then Green's Theorem gives

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy = 0$$

no matter what values P and Q assume in the region D.

Another application of the reverse direction of Green's Theorem is in computing areas. Since the area of D is $\iint_D 1\,dA$, we wish to choose P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

There are several possibilities:

$$P(x,y) = 0 \qquad P(x,y) = -y \qquad P(x,y) = -\frac{1}{2}y$$
$$Q(x,y) = x \qquad Q(x,y) = 0 \qquad Q(x,y) = \frac{1}{2}x$$

Then Green's Theorem gives the following formulas for the area of D:

(5) $$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

Example 3 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Soln: The ellipse has parametric equations $x = a\cos t$ and $y = b\sin t$, where $0 \le t \le 2\pi$. using the third formula in Eqn 5, we have

$$A = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt$$
$$= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab$$

Surface integrals:
The relationship between surface integrals and surface area much the same as the relationship between line integrals and arc length. Suppose f is a function of three variables whose domain includes a surface S. we divide S into patches S_{ij} with area ΔS_{ij}. we evaluate f at a point P^*_{ij} in each patch, multiply by the area ΔS_{ij} and form the sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P^*_{ij}) \Delta S_{ij}.$$

Then we take the limit as the patch size approaches 0 and define the surface integral of f over the surface S as

(1)
$$\iint_S f(x,y,z)\, dS = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P^*_{ij}) \Delta S_{ij}$$

Graphs: If the surface S is a graph of a function of two variables, then it has an equation of the form
$z = g(x,y)$, $(x,y) \in D$.
In fact, it can be shown that, if f is continuous on S and g has continuous derivatives, then Defn. 1 becomes

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y)) \sqrt{\{g_x(x,y)\}^2 + \{g_y(x,y)\}^2 + 1}\; dA$$

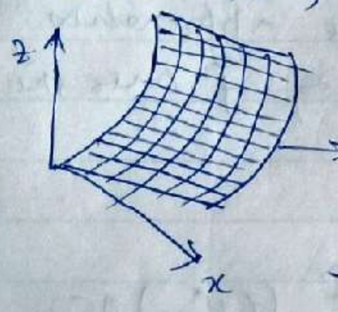
This formula is true even when D is not a rectangle and is usually written in the following form

2 $$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA$$

Similar formulas apply when it is more convenient to project S onto the yz plane or xz plane. For instance, if S is a surface with equation $y = h(x,z)$ and D is its projection on the xz-plane, then

$$\iint_S f(x,y,z)\,dS = \iint_D f(x, h(x,z), z)\sqrt{\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2+1}\,dA$$

Example 1 Evaluate $\iint_S y\,dS$, where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$ (see fig 2)



Soln: Since $\frac{\partial z}{\partial x} = 1$ and $\frac{\partial z}{\partial y} = 2y$

Formula 2 gives

$$\iint_S y\,dS = \iint_D y\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \int_0^1\int_0^2 y\sqrt{1+1+4y^2}\,dy\,dx$$

$$= \int_0^1 dx\,\sqrt{2}\int_0^2 y\sqrt{1+2y^2}\,dy$$

$$= \sqrt{2}\left(\frac{1}{4}\right)\frac{2}{3}(1+2y^2)^{3/2}\Big]_0^2 = \frac{13\sqrt{2}}{3}$$

Parametric surfaces

Suppose that a surface S has a vector equation

$$r(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}, \quad (u,v) \in D$$

3 $$\iint_S f(x,y,z)\,dS = \iint_D f(r(u,v))\,|r_u \times r_v|\,dA$$

Any surface S with equation $z = g(x,y)$ can be regarded as a parametric surface with parametric equation

$$x = x, \quad y = y, \quad z = g(x,y)$$

Example 2 Compute the surface integral $\iint_S x^2 dS$, where S is the unit sphere $x^2+y^2+z^2=1$

Soln: We use the parametric representation

$$x = \sin\varphi\cos\theta, \quad y = \sin\varphi\sin\theta, \quad z = \cos\varphi,$$

$$\cancel{0 \le \pi =} 0 \le \varphi \le \pi, \quad 0 \le \theta \le 2\pi$$

~~We compute $|r_\varphi \times r_\theta| = \sin\varphi$~~

that is $r(\varphi,\theta) = \sin\varphi\cos\theta\,\hat{i} + \sin\varphi\sin\theta\,\hat{j} + \cos\varphi\,\hat{k}$

Defn 6 If a smooth parametric surface S is given by the eqns $r(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$, $(u,v) \in D$ and S is covered just once as (u,v) ranges throughout parameter domain D, then the surface area of S is $A(S) = \iint_D |r_u \times r_v|\, dA$

where $r_u = \frac{\partial x}{\partial u} i + \frac{\partial y}{\partial u} j + \frac{\partial z}{\partial u} k$, $r_v = \frac{\partial x}{\partial v} i + \frac{\partial y}{\partial v} j + \frac{\partial z}{\partial v} k$

Find the surface area of a sphere of radius a

Parametric representation

$$x = a\sin\varphi\cos\theta, \quad y = a\sin\varphi\sin\theta, \quad z = a\cos\varphi$$

where the parameter domain is

$$D = \{(\varphi,\theta) ; 0 \le \pi \le \pi, \; 0 \le \theta \le 2\pi\}$$

$$r_\varphi \times r_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = a^2\sin^2\varphi\cos\theta\,\hat{i} + a^2\sin^2\varphi\sin\theta\,\hat{j} + a^2\sin\varphi\cos\varphi\, k$$

Teacher's Signature

(34)

we can compute that $|r_\varphi \times r_\theta| = \sin\varphi$

Therefore, by Formula 3,

$$\iint_S x^2\,dS = \iint_D (\sin\varphi\cos\theta)^2 |r_\varphi \times r_\theta|\,dA$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^2\varphi\cos^2\theta\sin\varphi\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\cos^2\theta\,d\theta \int_0^{\pi}\sin^3\varphi\,d\varphi = \frac{4\pi}{3}.$$

Example 3 Evaluate $\iint z\,dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2+y^2=1$, whose bottom S_2 is the disk $x^2+y^2\le 1$ in the plane $z=0$, and whose top S_3 is the part of the plane $z=1+x$ that lies above S_2.

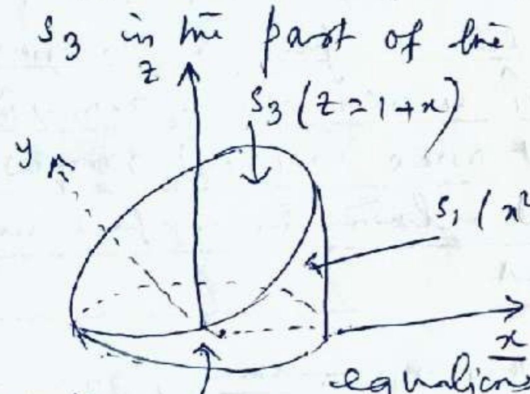


Fig. 4

Soln: The surface S is shown in Fig. 4. (we have changed the usual position of the axes to get a better look at S). For S_1 we use θ and z as parameters and write its parametric equations as

$$x=\cos\theta,\; y=\sin\theta,\; z=z$$

where $0\le\theta\le 2\pi$, and $0\le z\le 1+x = 1+\cos\theta$

Therefore

$$|r_\theta \times r_z| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

and $|r_\theta \times r_z| = \sqrt{\cos^2\theta+\sin^2\theta} = 1$

Thus, the surface integral over S_1 is

$$\iint_{S_1} z\,dS = \iint_D z\,|r_\theta \times r_z|\,dA$$

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta = \int_0^{2\pi}\frac{1}{2}(1+\cos\theta)^2\,d\theta = \frac{3\pi}{2}$$

Since S_2 lies in the plane $z=0$, we have

$$\iint_{S_2} z\,dS = \iint_{S_2} 0\,dS = 0$$

The top surface S_3 lies above the unit disk D and is part of the plane $z = 1+x$. So, taking $g(x,y) = 1+x$ in Formula 2 and converting to polar coordinates, we have

$$\iint_{S_3} z\,dS = \iint_D (1+x)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \int_0^{2\pi}\int_0^1 (1+r\cos\theta)\sqrt{1+1+0}\;r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 (r+r^2\cos\theta)\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{3}\cos\theta\right)d\theta = \sqrt{2}\,\pi$$

Therefore $\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS$

$$= \frac{3\pi}{2} + 0 + \sqrt{2}\pi = \left(\frac{3}{2}+\sqrt{2}\right)\pi$$

Stokes' Theorem

Stokes' theorem can be regarded as a higher-dimensional version of Green's theorem. Whereas Green's theorem relates a double integral over a plane region D to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve). F81 shows an oriented surface with unit normal vector $\hat{n}$.

The orientation of S induces the positive orientation of the boundary curve C shown in the figure. This means that if you walk in the positive direction around C with your head pointing in the direction of $\hat{n}$, then the surface will always be on your left.

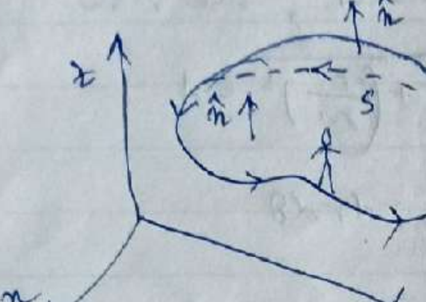


Fig.1.

Stokes Theorem:

Let S be an oriented piecewise-smooth surface that is bounded by a simple, piecewise-smooth boundary curve C with positive orientation. Let F be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C F \cdot dr = \iint_S \text{curl}\, F \cdot dS$$

In fact, in the special case where the surface S is flat and lies in the xy plane with upward orientation, the unit normal is $\hat{k}$, the surface integral becomes a double integral, and Stokes Theorem becomes

$$\int_C F \cdot dr = \iint_S \text{curl} F \cdot dS = \iint_S (\text{curl} F) \cdot \hat{k}\, dA$$

This is precisely the vector form of Green's theorem. Thus, we see that Green's theorem is really a special case of Stokes' theorem.

Although Stokes' Theorem is too difficult for us to prove in its full generality, we can give a proof when S

is a graph and F, S and C are well behaved.

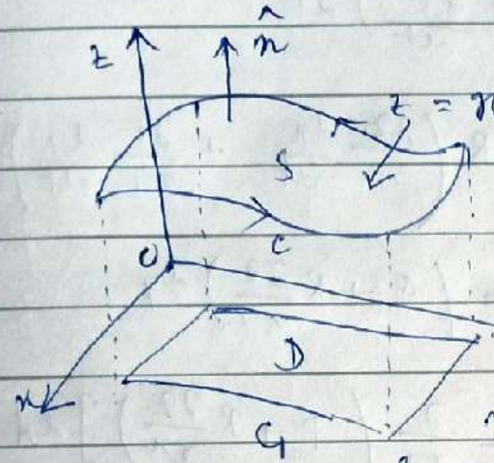


Fig 2.

Proof of a special case of Stokes' Theorem. We assume that the equation of S is $z = g(x,y)$, $(x,y) \in D$, where g has continuous second-order partial derivatives and D is a simple plane region whose boundary curve C_1 corresponds to C. If the orientation of S is (~~see Fig~~ 2) upward, then the positive orientation of C corresponds to the positive orientation of C_1 (see Fig. 2). We are given that $F = P\hat{i} + Q\hat{j} + R\hat{k}$, where the partial derivatives of P, Q, and R are continuous.

Since S is a graph of a function, we can apply formula
$$\iint_S F.\,dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$
with F replaced by curl F. The result is

$$2 \quad \iint_S \text{curl}\,F.\,dS = \iint_D \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dA$$

where the partial derivatives of P, Q and R are evaluated at $(x, y, g(x,y))$. If $x = x(t)$, $y = y(t)$, $a \le t \le b$ is a parametric representation of C_1, then a parametric representation of C is
$$x = x(t),\ y = y(t),\ z = g(x(t), y(t)) \quad a \le t \le b$$

This allows us, with the aid of the Chain Rule, to evaluate the line integral as follows:

$$\int_C F.dr = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) dt$$

$$= \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \right) \right] dt$$

$$= \int_a^b \left(P + R\frac{\partial z}{\partial x} \right) dx + \left(Q + R\frac{\partial z}{\partial y} \right) dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x} \right) \right] dA$$

where we have used Green's Theorem in the last step. Then, using the Chain Rule again and remembering that P, Q and R are functions of x, y and z and that z is itself a function of x and y, we get

$$\int_C F.dr = \iint_D \left[\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\cdot\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x \partial y} \right) \right.$$

$$\left. - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + \frac{R\partial^2 z}{\partial y \partial x} \right) \right] dA$$

Four of the terms in this double integral cancel and remaining six terms can be arranged to coincide with the right side of Eqn. 2. Therefore

$$\int_C F.dr = \iint_S \text{curl} F . dS.$$

<u>Example I</u> Evaluate $\int_C F.dr$ where $F(x,y,z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ and C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$ (Orient C to be counterclockwise

when viewed from above).

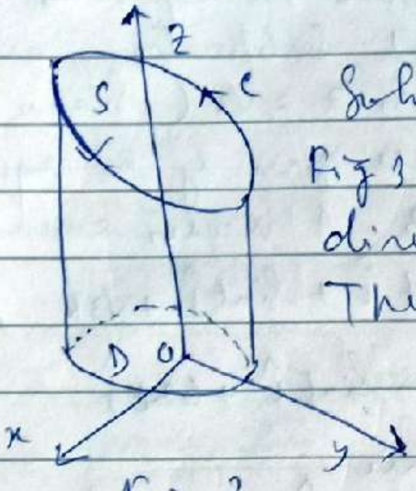


Fig. 3.

Soln: The curve C (an ellipse) is shown in Fig 3. Although $\int_C F \cdot dr$ could be evaluated directly, it is easier to use Stokes' Theorem. We first compute

$$\operatorname{curl} F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1+2y)\hat{k}$$

Although there are many surfaces with boundary C, the most convenient choice is the elliptical region S in the plane $y + z = 2$ that is bounded by C. If we orient S upward, then C has the induced positive orientation. The projection D of S on the xy plane is the disk $x^2+y^2 \le 1$ and so using equation $\iint_S F \cdot dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$

with $z = g(x,y) = 2 - y$ we have

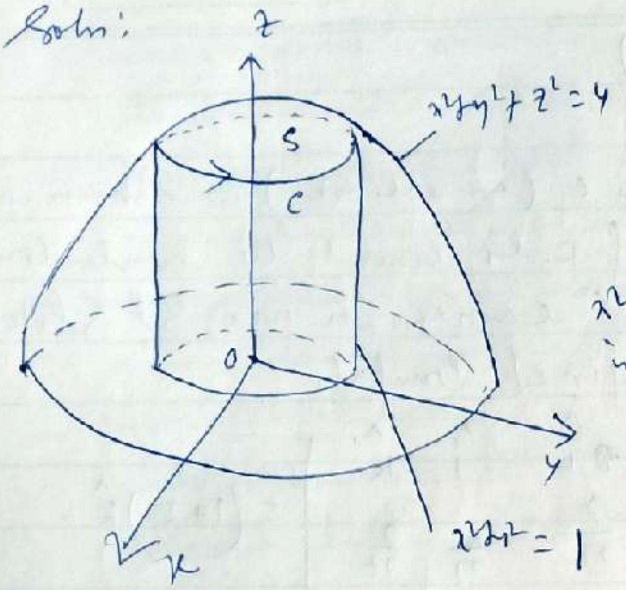
$$\int_C F \cdot dr = \iint_S \operatorname{curl} F \cdot dS = \iint_D (1+2y)\, dA$$

$$= \int_0^{2\pi}\int_0^1 (1 + 2r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + \frac{2r^3}{3}\sin\theta\right] d\theta = \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3}\sin\theta\right) d\theta = \pi$$

Example 2 Use Stokes' Theorem to compute the integral $\iint \operatorname{curl} F \cdot dS$ where $F(x,y,z) = xz\,\hat{i} + yz\,\hat{j} + xy\,\hat{k}$ and S is the part of the sphere $x^2+y^2+z^2 = 4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane (See Fig 8.4)

Soln:



To find the boundary curve C we solve the equations $x^2+y^2+z^2=4$ and $x^2+y^2=1$. Subtracting, we get $z^2=3$ and so $z=\sqrt{3}$ (since $z>0$). Thus, C is the circle given by the equations $x^2+y^2=1$, $z=\sqrt{3}$. A vector equation of C is $r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + \sqrt{3}k$, $0 \le t \le 2\pi$

So $r'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$

Also, we have

$F(r(t)) = \sqrt{3}\cos t\,\hat{i} + \sqrt{3}\sin t\,\hat{j} + \cos t \sin t\,\hat{k}$

Therefore, by Stokes' Theorem

$$\iint_S \text{curl} F \cdot dS = \int_C F\cdot dr = \int_0^{2\pi} F(r(t))\cdot r'(t)\,dt$$

$$= \int_0^{2\pi} (-\sqrt{3}\cos t \sin t + \sqrt{3}\sin t \cos t)\,dt = \sqrt{3}\int_0^{2\pi} 0\,dt = 0$$

The Divergence Theorem

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let F be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S F\cdot dS = \iiint_E \text{div} F\,dV$$

Proof. Let $F = P\hat{i} + Q\hat{j} + R\hat{k}$. Then

$$\text{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

So $$\iiint_E \text{div} F\,dV = \iiint_E \frac{\partial P}{\partial x}\,dV + \iiint_E \frac{\partial Q}{\partial y}\,dV + \iiint_E \frac{\partial R}{\partial z}\,dV$$