VIVEKANANDA COLLEGE, THAKURPUKUR

KOLKATA-700063

NAAC ACCREDITED ‘A’ GRADE

Topic :	Real Analysis
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Name of the Teacher :	Chandan Maji
Name of the Department:	Mathematics

Sequence of real nos.

Defn: A mapping $f: \mathbb{N} \to \mathbb{R}$ is said to be a sequence in $\mathbb{R}$ or a real sequence & its f-image $f(1), f(2), \ldots f(n), \ldots$ are real nos. The image of the nth element $f(n)$ is said to be the nth element of the real seq.

A sequence 'f' is generally denoted by $\{f(n)\}_n$.

The range of the sequence $\{f(n)\}$ is a subset of $\mathbb{R}$.

Examp: ① let $f: \mathbb{N} \to \mathbb{R}$ by $f(n) = n$, $n \in \mathbb{N}$

$f(1) = 1,\ f(2) = 2,\ f(3) = 3, \ldots$

$\{n\} = \{1, 2, 3, \ldots\}$.

(2) let $f: \mathbb{N} \to \mathbb{R}$ by $f(n) = n^2$, $n \in \mathbb{N}$.

Sequence is denoted by $\{n^2\} = \{1^2, 2^2, 3^2, \ldots\}$.

(3) let $f: \mathbb{N} \to \mathbb{R}$ by $f(n) = \frac{1}{n}$, $n \in \mathbb{N}$.

Sequence is $\{1/n\} = \{1, 1/2, 1/3, \ldots\}$.

(4) let $f: \mathbb{N} \to \mathbb{R}$ by $f(n) = (-1)^n$, $n \in \mathbb{N}$.

$\{(-1)^n\} = \{-1, 1, -1, 1, \ldots\}$.

Range $= \{-1, 1\}$.

(5) let $f: \mathbb{N} \to \mathbb{R}$ by $f(n) = 2$, $\forall n \in \mathbb{N}$.

$\{f(n)\} = \{2, 2, \ldots\}$ Constant Sequence.

Bounded Sequences: A sequence $\{f(n)\}$ is said to be bounded above if there exist a real number G s.t $f(n) \le G$. $\forall n \in \mathbb{N}$. G is said to be an upper bound of the sequence.

A real sequence $\{f(n)\}$ is said to be bounded below if there exist a real number g s.t $f(n) \geq g$ $\forall n \in \mathbb{N}$. g is said to be a lower bound of the sequence.

Def^n: A Sequence $\{f(n)\}$ is said to be bounded sequence if there exist real nos. G, g such that $g \leq f(n) \leq G$. $\forall n \in \mathbb{N}$.

Supremum / least upper bound: (sup / LUB)

The least upper bound of a real sequence $\{f(n)\}$ is a real no. M satisfying the following conditions:

(i) $f(n) \leq M$ $\forall n \in \mathbb{N}$

(ii) for each pre assigned positive ϵ, $\exists$ a natural no. k s.t $f(k) > M - \epsilon$.

So, we can write $\sup \{f(n)\} = M$.

Infimum / greatest lower bound: (Inf / GLB)

The greatest lower bound of a real sequence $\{f(n)\}$ is a real no. m satisfying the following cond.

(i) $f(n) \geq m$ $\forall n \in \mathbb{N}$

(ii) for each pre assigned +ve ϵ $\exists$ a natural no k s.t $f(k) < m + \epsilon$.

So, $\inf \{f(n)\} = m$.

Note: For a real sequence $\{f(n)\}$ unbounded above we define $\sup \{f(n)\} = \infty$.

and for a real sequence $\{f(n)\}$ unbounded below, we define $\inf \{f(n)\} = -\infty$.

Example: For the following sequence

(i) $f(n) = \{1/n\}$ — bounded seq. $\sup (f(n)) = 1$

$\inf f(n) = 0$.

Sequence	Sup	Inf	Nature	Conv/div
$\{1/n\}$	1	0	Bounded Sequence	conv
$\{n^2\}$	∞	1	Bounded below but unbounded above	div
$\{-2n\}$	-2	$-\infty$	Bounded above but unbounded below	div
$\{(-1)^n . n\}$	∞	$-\infty$	unbounded above & unbounded below	div
$\{(-1)^n\}$	1	-1	bounded & oscillatory seq.	oscillatory

Limit of a Sequence:

A real no. l is said to be the limit of the sequence $\{f(n)\}$ if corresponding to a pre-assigned (+)ve ϵ there exist a natural no K. (depending on ϵ) s.t

$$|f(n) - l| < \epsilon \quad \forall\ n \geq K$$

$$\text{ie} \quad l - \epsilon < f(n) < l + \epsilon \quad \forall\ n \geq K.$$

Note: A real sequence can have at most one limit.

Convergent Sequence:

A real sequence $\{f(n)\}$ is said to be a convergent sequence if it has a limit $l \in R$ & in this case the sequence is said to converge to l. & we may write.

$$\lim_{n \to \infty} f(n) = l. \text{ or } \lim f(n) = l.$$

Limit theorems: let $\{u_n\}$ & $\{v_n\}$ be two convergent sequences that converges to u & v respectively, then (i) $\lim (u_n + v_n) = u + v$.

(ii) $\lim (c.u_n) = c. \lim u_n$ if $c \in R$.

(iii) $\lim u_n/v_n = \frac{u}{v}$, provided $\{v_n\}$ is a sequence of non zero real nos & $v \neq 0$.

Note: A constant sequence is a convergent sequence.

Ex. Show that a convergent sequence is bounded. Is the converse true? Justify.

→ Let, $\{f(n)\}$ be a convergent sequence & let l be its limit. Let us choose $\epsilon = 1$. For this chosen ϵ there exist a natural no. k s.t

$$l-1 < f(n) < l+1 \quad \forall n \geq k.$$

Let $B = \max\{f(1), f(2), \dots f(k-1), l+1\}$

$b = \min\{f(1), f(2), \dots f(k-1), l-1\}$

then $b \leq f(n) \leq B \quad \forall n \in \mathbb{N}$.

ie the sequence $\{f(n)\}$ is bounded.

Converse of the theorem is always not true ie a bounded sequence may not be a a convergent sequence.

For example, the sequence $\{(-1)^n\}$ is a bounded seq but the sequence does not converge to a limit.

Ex Let $\{u_n\}$ be a convergent sequence of real nos. Converging to u. Then s.t the sequence $\{|u_n|\}$ converges to $|u|$. Converse is true? When it will be true?

→ We have $||u_n| - |u|| \leq |u_n - u|$. —— (1)

Let $\epsilon > 0$ & $\lim\limits_{n\to\infty} u_n = u$ (given). So $\exists$ a natural no. k s.t $|u_n - u| < \epsilon \quad \forall n \geq k$

Now from (1),

$$||u_n| - |u|| \leq |u_n - u| < \epsilon \quad \forall n \geq k.$$

ie $||u_n| - |u|| < \epsilon \quad \forall n \geq k.$

Since, ϵ is arbitrary, $\lim |u_n| = |u|$.

Converse, of the theorem is not true ie if $\{|u_n|\}$ is a convergent seq ~~con~~ it does not necessarily imply that $\{u_n\}$ is a convergent seq.

for example, let $u_n = (-1)^n$.

Then the seq $\{|u_n|\}$ converges to 1.

but the seq $\{u_n\}$ is not convergent.

Converse will be true when the limit of the seq $u = 0$.

Null Sequence: A sequence $\{u_n\}$ is said to be a null sequence if $\lim u_n = 0$.

Divergent Seq: A seq $\{f(n)\}$ is said to be diverge to ∞ if corresponding to a pre-assigned positive number G, however large, there exist a natural no. k s.t

$$f(n) > G \quad \forall \; n \geq k.$$

& we write, $\lim_{n \to \infty} f(n) = \infty$.

Example: (i) $\{2^n\}$ diverges to ∞

(ii) $\{-n^2\}$ diverges to $-\infty$.

(iii) $\{(-1)^n\}$ is a bounded seq, but not convergent. It is an oscillatory seq of finite oscillation

(iv) $\{(-1)^n n\}$ is an unbounded seq, & it is not properly divergent. it is an oscillatory seq. of infinite oscillation.

Note: ~~An~~ oscillatory sequence is therefore neither convergent nor properly divergent. It is called an improperly divergent sequence..

Monotone Sequence:

A real seq $\{f(n)\}$ is said to be monotone increasing sequence if $f(n+1) \geq f(n)$ $\forall\, n \in \mathbb{N}$ & a real seq $\{f(n)\}$ is said to be monotone decreasing sequence if $f(n+1) \leq f(n)$ $\forall\, n \in \mathbb{N}$.

A real seq $\{f(n)\}$ is said to be a monotone seq if it is either a monotone increasing seq or a monotone decreasing seq.

Note: If $f(n+1) > f(n)$ $\forall\, n \in \mathbb{N}$ then the seq $\{f(n)\}$ is strictly monotone increasing seq & if $f(n+1) < f(n)$ $\forall\, n \in \mathbb{N}$ then the seq $\{f(n)\}$ is strictly monotone decreasing seq.

Example: (i) let $f(n) = 2^n$ $\forall\, n \geq 1$.

$f(n+1) = 2^{n+1} > f(n)$.

$\Rightarrow \{f(n)\}$ — strictly m.i.

(ii) let, $f(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$, $\forall\, n \geq 1$.

$f(n+1) - f(n) = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$.

$= \frac{1}{(2n+1)(2n+2)} > 0$ $\forall\, n \in \mathbb{N}$

$\Rightarrow \{f(n)\}$ — strictly m.i.

(iii) let, $f(n) = \frac{1}{n}$, $\forall\, n \geq 1$

$f(n+1) - f(n) = \frac{1}{n+1} - \frac{1}{n} < 0$ $\forall\, n \in \mathbb{N}$.

$\Rightarrow \{f(n)\} \rightarrow$ strictly m.d.

(iv) let, $\{(-2)^n\}$, neither a m.i or m.d seq.

So, it is not a monotone seq.

Sequence (Set-2)

Ex. S.T a monotone increasing sequence if bounded above is convergent & it converges to the least upper bound or Sup.

⇒ let, $\{f(n)\}$ be a m.i sequence bounded above & let M be its least upper bound.

then (i) $f(n) \le M \quad \forall n \in \mathbb{N}$

& (ii) for a pre-assigned (+)ve ϵ, there exist a natural no. k st $f(k) > M - \epsilon$.

Since $\{f(n)\}$ is a monotone increasing seq.

so, $M - \epsilon < f(k) \le f(k+1) \le f(k+2) \le \cdots \le M$.

i.e $M - \epsilon < f(n) < M + \epsilon \quad \forall n \ge k$.

$\therefore$ $\{f(n)\}$ is convergent & converge to M

i.e ~~Sup $\{f(n)\}$~~ $\lim f(n) = M$.

Ex. S.T a monotone decreasing sequence, if bounded below is convergent & converges to the greatest lower bound.

Ex. Show that the following sequence $\{(1+1/n)^n\}$ is a (i) monotone increasing seq (ii) bounded above (iii) convergent & find its limit.

⇒ Let, $u_n = (1 + 1/n)^n$. then $u_{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1}$.

let us consider $(n+1)$ positive numbers

$1 + \frac{1}{n}, 1 + \frac{1}{n}, \cdots 1 + \frac{1}{n}$ (n times) & 1.

Now applying AM > GM we have

$$\frac{n(1+1/n)+1}{n+1} > \left(1 + \frac{1}{n}\right)^{n/n+1}$$

or $$\left(1 + \frac{1}{n+1}\right)^{n+1} > \left(1 + \frac{1}{n}\right)^n.$$

ie $u_{n+1} > u_n \quad \forall n \in \mathbb{N}$.

$\therefore$ The sequence $\{u_n\}$ is a monotone increasing sequence.

Now $u_n = 1+1+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\cdots+\frac{n(n-1)\cdots 2\cdot 1}{n!}\frac{1}{n^n}$.

$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\frac{2}{n}\cdot\frac{1}{n}$

$< 1+1+\frac{1}{2!}+\cdots+\frac{1}{n!} \quad \forall\, n \geq 2$

Now we have, $n! > 2^{n-1}$. $\forall\, n > 2$.

Utilising this,

$1+1+\frac{1}{2!}+\cdots+\frac{1}{n!} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} \quad \forall\, n \geq 2$

$= 1+2\left[1-\left(\frac{1}{2}\right)^n\right] < 3. \;\forall\, n \in \mathbb{N}$

It follows that $u_n < 3 \;\forall\, n \in \mathbb{N}$.

Hence, the sequence is bounded above by 3.

$\therefore$ $\{u_n\}$ is a monotone increasing sequence & bounded above, hence it is convergent.

the limit of the seq is denoted by e

Since $u_1 = 2$ it follows that $2 < u_n < 3$. $\forall\, n \geq 2$.

Ex. Show that, the seq $\{x_n\}$, where $x_n = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}$ is a monotone increasing sequence, bounded above and $\lim x_n = e$.

Ex. Show that, the seq $\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}$ is a monotone decreasing seq with limit e.

Two important properties

(A) Let, $\{u_n\}$ be a sequence of (+)ve real numbers such that $\lim \frac{u_{n+1}}{u_n} = l$. then

(i) If $0 \le l < 1$ then $\lim u_n = 0$

(ii) If $l > 1$ then $\lim u_n = \infty$

(B) Let, $\{u_n\}$ be a sequence of (+)ve real nos. such that, $\lim \sqrt[n]{u_n} = l$. then

(i) If $0 \le l < 1$ then $\lim u_n = 0$

(ii) If $l > 1$ then $\lim u_n = \infty$.

Ex. A sequence $\{u_n\}$ is defined by $u_{n+2} = \frac{1}{2}(u_{n+1} + u_n)$ for $n \geq 1$. and $0 < u_1 < u_2$. Prove that the sequence $\{u_n\}$ converges to $\frac{u_1 + 2u_2}{3}$.

$\Rightarrow$ Given, $0 < u_1 < u_2$ ie $u_2 - u_1 > 0$.

Now, $u_3 - u_2 = \frac{1}{2}(u_2 + u_1) - u_2 = -\frac{1}{2}(u_2 - u_1)$

$u_4 - u_3 = \frac{1}{2}(u_3 + u_2) - u_3 = \frac{1}{2}(u_2 - u_3)$

$= \left(-\frac{1}{2}\right)^2 (u_2 - u_1)$

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$u_n - u_{n-1} = \left(-\frac{1}{2}\right)^{n-2} (u_2 - u_1)$

Therefore, $u_n - u_1 = (u_2 - u_1)\left[1 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^2 + \cdots + \left(-\frac{1}{2}\right)^{n-2}\right]$.

$= \frac{2(u_2 - u_1)}{3}\left[1 - \left(-\frac{1}{2}\right)^{n-1}\right]$.

Now taking limit on both sides we get,

$\lim (u_n - u_1) = \frac{2}{3}(u_2 - u_1)$ as $\lim_{n \to \infty} \left(-\frac{1}{2}\right)^{n-1} = 0$

$\therefore \lim u_n = u_1 + \frac{2}{3}(u_2 - u_1)$

$= \frac{u_1 + 2u_2}{3}$. (Proved).

Ex. A sequence $\{u_n\}$ is defined by $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2u_1}$ for $n \geq 1$. prove that $\lim u_n = 2$.

$\rightarrow$ $u_1 = 2^{1/2}$, $u_2 = \sqrt{2\sqrt{2}} = 2^{\frac{1}{2}+\frac{1}{2^2}} = 2^{1-\frac{1}{2^2}}$

$u_3 = 2^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}} = 2^{1-\frac{1}{2^3}}$

$\vdots$

$u_n = 2^{1-\frac{1}{2^n}}$.

$\lim u_n = \lim 2^{1-\frac{1}{2^n}} = \lim 2^{x_n}$.

where $x_n = 1 - \frac{1}{2^n}$ (let)

$\lim x_n = 1$.

$\therefore \lim u_n = \lim 2^{x_n} = 2$, since

$\lim x_n = l$ & $a > 0 \Rightarrow \lim a^{x_n} = a^l$.

Ex. show that a m.i seq that is unbounded above diverges to ∞.

$\rightarrow$ let $\{f(n)\}$ be a monotone increasing seq and unbounded above ie not bounded above.

Since the seq is unbounded above so for a pre-assigned positive number G (however large) there exist a natural no k such that $f(k) > G$.

Since the seq $\{f(n)\}$ is m.i. so,

$G < f(k) \leq f(k+1) \leq f(k+2) \leq \cdots$

ie $f(n) > G \quad \forall \; n \geq k$.

ie the seq $\{f(n)\}$ diverges to ∞.

Ex. show that a m.d sequence ie unbounded below diverges to $-\infty$.

Cantor's theorem on nested intervals:

Let, $\{[a_n, b_n]\}$ be a sequence of closed and bounded intervals such that,

(i) $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n \in \mathbb{N}$ and

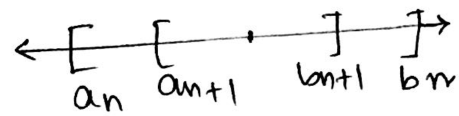
(ii) $\lim \delta_n = 0$ where $\delta_n = b_n - a_n$ = length of $[a_n, b_n]$.

Then show that $\bigcap_{n=1}^{\infty} [a_n, b_n]$ contains precisely one point.

→ Given, $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$

So, $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ $\forall n \in \mathbb{N}$.

Also, $a_n \leq b_n$ $\forall n \in \mathbb{N}$



Therefore we get,

$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1$

This shows that the sequence $\{a_n\}$ is a m.i sequence and bounded above & the seq $\{b_n\}$ is a m.d seq & bounded below.

Hence both the seq convergent.

let $\lim a_n = l$ & $\lim b_n = m$.

Since $\lim (b_n - a_n) = 0$. So, $l = m = \alpha$ (let).

Therefore α is the least upperbound of the seq $\{a_n\}$ & the greatest lower bound of the seq $\{b_n\}$.

Hence $a_n \leq \alpha$ & $\alpha \leq b_n$ $\forall n \in \mathbb{N}$.

ie $a_n \leq \alpha \leq b_n$

ie $\alpha \in [a_n, b_n]$. $\forall n \in \mathbb{N}$.

So, $\alpha \in \bigcap_{n=1}^{\infty} [a_n, b_n]$

Now we want to prove that α is the only point in $\bigcap_{n=1}^{\infty} [a_n, b_n]$.

If possible let, $\beta \in \bigcap_{n=1}^{\infty} [a_n, b_n]$.

Then $a_n \leq \beta \leq b_n$ $\forall n \in \mathbb{N}$.

Now define a sequence $\{u_n\}$ by $u_n = \beta$ $\forall n \in \mathbb{N}$

Then $\lim u_n = \beta$.

now $a_n \le u_n \le b_n \quad \forall\ n \ge 1$

and $\lim a_n = \alpha = \lim b_n$.

By Sandwich theorem $\lim u_n = \alpha$ & so, $\alpha = \beta$.

i.e. α is unique.

$\boxed{\text{Very imp Note:}}$ The theorem Says that a nested seq of closed and bounded intervals has a non empty intersection.

Now the question is: Whether a nested seq of <u>open & bounded</u> intervals have a non empty intersection ??

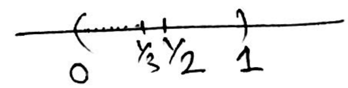
the answer is No. It may not have a non empty intersection.

For example: let $I_n = \{x \in \mathbb{R} : 0 < x < 1/n\}$ $\forall n \in \mathbb{N}$.

then $\{I_n\}$ is a nested seq of open bounded intervals as

$I_{n+1} \subset I_n$. $\forall\ n \in \mathbb{N}$

but here $\bigcap_{n=1}^{\infty} I_n = \phi$



$I_1 = (0,1)$

$I_2 = (0, 1/2)$

Next question: whether a nested seq of <u>closed and unbounded</u> intervals have a non empty intersection.

The answer is No.

For example: let $I_n = \{x \in \mathbb{R} : x \ge n\}$ $\forall n \in \mathbb{N}$

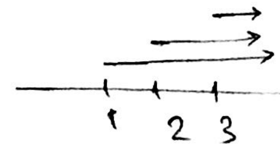
Then $\{I_n\}$ is a ~~closed~~ nested seq of closed and unbounded intervals as $I_{n+1} \subset I_n$. $\forall\ n \in \mathbb{N}$.

But here $\bigcap_{n=1}^{\infty} I_n = \phi$.

$I_1 = x \ge 1$

$I_2 = x \ge 2$

$I_3 = x \ge 3$



~~B~~.

Ex. Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2u_n}$ for all $n \geq 1$ converges to 2.

$\to$ The given sequence is $\{u_n\} = \left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\right\}$

Here, $u_{n+1} = \sqrt{2u_n}$. and $u_n = \sqrt{2u_{n-1}}$

$u_{n+1}^2 = 2u_n$. and $u_n^2 = 2u_{n-1}$.

Now, $u_{n+1}^2 - u_n^2 = 2(u_n - u_{n-1})$

$\Rightarrow (u_{n+1} + u_n)(u_{n+1} - u_n) = 2(u_n - u_{n-1})$

since $u_n > 0$. $\forall n \in \mathbb{N}$.

So, $u_{n+1} >$ or $< u_n$ according as $u_n >$ or $< u_{n-1}$.

But $u_1 < u_2$

Consequently, $u_2 < u_3$ & $u_3 < u_4, \dots$

and therefore $\{u_n\}$ is a monotone increasing Sequence.

Again, $2u_n = u_{n+1}^2 > u_n^2$ $\forall n \in \mathbb{N}$.

ie $u_n^2 - 2u_n < 0$ $\forall n \in \mathbb{N}$

or $u_n(u_n - 2) < 0$ $\forall n \in \mathbb{N}$.

but as $u_n > 0$, so $u_n < 2$ $\forall n \in \mathbb{N}$.

This shows that the sequence $\{u_n\}$ is bounded above and hence it is convergent.

let $\lim u_n = l$.

by def $u_{n+1}^2 = 2u_n$ $\forall n \in \mathbb{N}$.

taking limit as $n \to \infty$ we have, $l^2 = 2l$,

Therefore l is either 0 or 2. But l cannot be 0 as the seq is m.i and $u_1 = \sqrt{2} > 1$.

therefore, $l = 2$ & hence the seq converges to 2.

Ex. Prove that the seq $\{u_n\}$ defined by $u_1 = \sqrt{7}$ and $u_{n+1} = \sqrt{7+u_n}$ $\forall\, n \geq 1$ converges to the positive root of the equation $x^2 - x - 7 = 0$.

$\Rightarrow$ The sequence is $\{\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \ldots\}$.

Again, $u_{n+1}^2 - u_n^2 = u_n - u_{n-1}$.

or $(u_{n+1} + u_n)(u_{n+1} - u_n) = u_n - u_{n-1}$.

Since $u_n > 0$ $\forall\, n \in \mathbb{N}$. So, $u_{n+1} > u_n$ or $< u_n$ according as $u_n >$ or $< u_{n-1}$.

But $u_2 > u_1$ & consequently, $u_3 > u_2, u_4 > u_3, \ldots$ & therefore $\{u_n\}$ is a m.i seq.

Again, $u_n^2 < u_{n+1}^2 = 7 + u_n$.

or $u_n^2 - u_n - 7 < 0$.

or $(u_n - \alpha)(u_n - \beta) < 0$ where α & β are the roots of the equation $x^2 - x - 7 = 0$. One of two roots must be negative and other is positive.

let, $\alpha < 0$.

Since $u_n > 0$ so, $u_n - \alpha > 0$.

Consequently, $u_n < \beta$ $\forall\, n \in \mathbb{N}$.

Therefore, the seq $\{u_n\}$ is bounded above & hence it is conv.

let $\lim u_n = l$. By def. $u_{n+1}^2 = 7 + u_n$ $\forall\, n \in \mathbb{N}$.

taking lt $n \to \infty$, so we have, $l^2 = 7 + l$.

Therefore $(l - \alpha)(l - \beta) = 0$.

but $l \neq \alpha$ since each element of the seq is positive & $\alpha < 0$. Therefore $l = \beta$.

ie the seq converges to the (+)ve root of the equation $x^2 - x - 7 = 0$.

Ref: Real Analysis by S.K. Mapa.