

Basic Concepts of Partial Differential Equations (Study Material)

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Study Material June 2018

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1 What are Partial Differential Equations?

Partial differential equations (PDE) arise in the area of geometry, physics and applied mathematics. It can be used to describe a wide variety of phenomena such as sound, heat, electrostatics, fluid dynamics and quantum mechanics. The key defining property of a partial differential equation is that there is more than one independent variable x, y, z, \dots . Any dependent variable 'u' will be a function of these variables $u(x, y, z, \dots)$. and hence it possesses partial derivatives with respect to several independent variables.

2 Preliminaries

• **Defination (PDE):** A differential equation involving partial derivatives of an unknown function of more than one independent variables is known as partial differential equation.

For example, if x and y are two independent variables and z be a dependent variable i.e z is a function of x and y then

$$(i) \frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2} \quad (ii) \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0 \quad (iii) x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}$$

are all partial differential equations.

• **Order of a PDE :** The order of a PDE is the order of highest partial derivatives(or derivatives) occurring in the differential equations.

Any first order partial differential equation with one dependent variable z and two independent variable x and y is of the form

$$F(x, y, z, p, q) = 0 \text{ --- (1)}$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

We may classify PDE of first order depending upon the form of the function F .

(i) If equation (1) is of the form

$$P(x, y, z) \frac{\partial z}{\partial x} + Q(x, y, z) \frac{\partial z}{\partial y} = R(x, y, z)$$

is a quasi-linear PDE of first order if the derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ that appears in the function F are linear, while the coefficients P, Q and R depend on x, y and also z .

For example, $P(z) \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0, \quad x e^z p + 3y z q = x y z$ are quasi-linear PDE of first order.

(ii) Similarly equation (1) if can be put of the form

$$P(x, y) \frac{\partial z}{\partial x} + Q(x, y) \frac{\partial z}{\partial y} = R(x, y, z)$$

then it is called almost linear linear PDE of first order, if the coefficients P and Q are functions of x and y only.

For example, $3xyp + 8yp = 5z^3x$.

(iii) Equation (1) is called a linear partial differential equation if it can be put of the form

$$a(x, y) \frac{\partial z}{\partial x} + b(x, y) \frac{\partial z}{\partial y} + C_1(x, y)z = C_2(x, y)$$

where F is linear in p,q and z and coefficients a, b, C_1, C_2 are depends only x and y.

For example, $px + qy = nz$, $x^2yp + y^2xq = x^2y^2z + e^x$ are of first order linear PDE.

- Any equation which can not fit into above these three catagories is called non-linear PDE. For example,

$$p^2 + q^2 = 1, \quad pq = 1, \quad p^2 + pq = ze^x.$$

3 Formation of Partial Differential Equations

Here we shall show that how PDE can be formulated. There are two methods by which one can formulate a first order PDE.

- **Method 1 (Elimination of arbitrary constants).**

Let us define a function, $f(x, y, z, a, b) = 0$ -----(2).

where a and b are arbitrary constants. Again consider z is a function of x and y.

Now differentiating (2) partially with respect to x and y, we get

$$\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} = 0$$
 -----(2.1)

$$\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} = 0,$$
 ----- (2.2)

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

Now eliminating a and b from (2), (2.1) and (2.2) we get a relation of the form

$$f(x, y, z, p, q) = 0$$

which is desired PDE of first order.

• **Method 2 (Derivation of a PDE by the elimination of arbitrary functions).**

Let, u and v are two functions of x, y, z and F be any arbitrary function is of the form

$$F(u, v) = 0 \text{ --- --- --- (3)}$$

again consider z is a function of x and y . We try to find a differential equation by eliminating the arbitrary function F .

Now partially differentiating (3) with respect to x and y we get,

$$\frac{\partial F}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} \right) + \frac{\partial F}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial x} \right) = 0$$

or, $\frac{\partial F}{\partial u} \left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right) + \frac{\partial F}{\partial v} \left(\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right) = 0$, and

$$\frac{\partial F}{\partial u} \left(\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} \right) + \frac{\partial F}{\partial v} \left(\frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z} \right) = 0$$

Now eliminate $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from this two equations and we get,

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} & \frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z} \end{vmatrix} = 0,$$

which simplifies to

$$pP + qQ = R \text{ --- --- --- (4)}$$

where $P = \frac{\partial(u,v)}{\partial(y,z)} = \frac{\partial u}{\partial y} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial y}$, $Q = \frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial u}{\partial z} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z}$, $R = \frac{\partial(u,v)}{\partial(y,x)} = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}$.

This is a linear PDE.

4 Examples:

• **Example 1.** Obtain a PDE by eliminating the arbitrary function from $z = f\left(\frac{xy}{z}\right)$.

Solution. Here given $z = f\left(\frac{xy}{z}\right)$.

Now differentiating partially with respect to x and y , we obtain

$$p = \frac{\partial z}{\partial x} = \frac{y}{z} f' \left(\frac{xy}{z} \right) \text{ and } q = \frac{\partial z}{\partial y} = \frac{x}{z} f' \left(\frac{xy}{z} \right).$$

Eliminating f' from these two equations, we get

$$xp = yq \text{ which is required PDE.}$$

• **Example 2.** Form the PDE by eliminating the arbitrary function from

$z = f(x + it) + g(x - it)$, where $i = \sqrt{-1}$.

Solution Given $z = f(x + it) + g(x - it)$ ————(1)

Differentiating (1) partially with respect to x and t , we get

$$\frac{\partial z}{\partial x} = f'(x + it) + g'(x - it)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x + it) + g''(x - it) \text{—————(2a)}$$

and

$$\frac{\partial z}{\partial t} = if'(x + it) - ig'(x - it)$$

$$\frac{\partial^2 z}{\partial t^2} = -f''(x + it) - g''(x - it) \text{—————(2b)}$$

From (2a) and (2b), we get $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$ which is required PDE.

• **Example 3.** Find the differential equation of the set of all right circular cones whose axes coincide with z axis.

Solution. General equation of the set of all right circular cones whose axes coincides with z -axis is given by

$$x^2 + y^2 = (z - a)^2 \tan^2 \alpha \text{—————(1)}$$

where α be the semivertical angle and $(0, 0, a)$ be the coordinates of the vertex. Here a and α are arbitrary constants.

Differentiating (1) partially with respect to x and y , we get

$$2x = 2(z - a) \frac{\partial z}{\partial x} \tan^2 \alpha$$

or,

$$(z - a) \tan^2 \alpha = \frac{x}{p}$$

and similarly,

$$(z - a) \tan^2 \alpha = \frac{y}{q}$$

Eliminating a and α from these two equations, we get

$$xq = yp$$

which is the required differential equation.

• **Bibliography.**

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