STUDY MATERIAL

VIVEKANANDA COLLEGE THAKURPUKUR

NAAC ACCREDITED GRADE-A

MATHEMATICS
(HONOURS)

COMPLEX NUMBERS

Dr. JAYEETA SAHA

ASSISTANT PROFESSOR
DEPARTMENT OF MATHEMATICS,
VIVEKANANDA COLLEGE, THAKURPUKUR

Complex Numbers

Any complex number z is an ordered pair (a, b) of two real numbers a and b such that
$(i)(a, b) = (c, d)$ if and only if $a = c$ and $b = d$
$(ii)(a, b) + (c, d) = (a + c, b + d)$
$(iii)(a, b) \cdot (c, d) = (ac - bd, ad + bc)$
where (c, d) is an another complex number. The first component a of (a, b) is called real part while the second component is called imaginary part of (a, b).

Also any complex number $z = (a, b)$ is commonly expressed as $a + ib$, where i is the complex number $(0, 1)$.If the real part of two complex numbers are same and their imaginary parts are also same but of opposite signs then the two numbers are said to be complex conjugate of each other.

Polar form of complex number

Let $z = a + ib$ be a complex number. Let us take two mutually perpendicular straight lines as real axis and imaginary axis. Taking their meet points as the pole and real axis as the initial line, let (r, θ) be the polar coordinates of the point (a, b).Then $a = r\cos\theta$ and $b = r\sin\theta$.

Geometrically r is the distance of the point (a, b) from the pole (origin) which is called modulus of the complex number and θ is the angle made by the radius vector through the point (a, b) with the real axis and is called an argument (amplitude) of z.

Therefore a non-zero coplex number $z = a + ib$ is represented in the form $z = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{(a^2 + b^2)}$ and $\theta = \tan^{-1}(\frac{b}{a})$ for $a > 0$ or $\theta = \tan^{-1}(\frac{b}{a}) + \pi$ for $a < 0$. This is called polar form or modulus-amplitude form of the complex number z.

Note 0.1. (i). As θ is indeterminate f0r the zero compex number, the zero complex number has no polar representation.

(ii). As we know that $\cos(\theta) = \cos(\theta + 2n\pi)$ and $\sin(\theta) = \sin(\theta + 2n\pi)$ for any integer n, so θ has infinitely many values. All values of θ are expressed as Arg$z$ (or Amp$z$). But the principal value of Arg$z$(Amp$z$), denoted by arg$z$(amp$z$) is defined to be the angle θ satisfying the relation $-\pi < \theta \leq \pi$.

Example 0.2. Find out mod$z$ and arg$z$, where $z = \frac{(1+i)(2+3i)}{(i-1)(2-3i)}$

$z = \frac{(1+i)(2+3i)}{(i-1)(2-3i)} = \frac{2+2i+3i-3}{2i-2+3+3i} = \frac{-1+5i}{1+5i} = \frac{-(1-5i)^2}{(1+5i)(1-5i)} = \frac{-1+10i+25}{1+25} = \frac{12}{13} + \frac{5}{13}i$

So, $|z| = \sqrt{\left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2} = 1$ and amp$z = tan^{-1}(\frac{5}{12})$.

Example 0.3. Express $(-1-i)$ in the polar form.

Let $-1-i = r(\cos\theta + isin\theta)$. Then $r\cos\theta = -1$ and $r\sin\theta = -1$.

So $r^2 = 2$, hence $r = \sqrt{2}$.

Also $\cos\theta = -\frac{1}{\sqrt{2}}$ and $\sin\theta = -\frac{1}{\sqrt{2}}$. These determine $\theta = \frac{5\pi}{4}$.

Hence $-1-i = \sqrt{2}(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4})$.

Note 0.4. Here $\theta = \frac{5\pi}{4}$ is not the principal argument. The principal argument is $-\frac{3\pi}{4}$. So in case of principal argumant the polar form of $-1-i$ is $\sqrt{2}[\cos(-\frac{3\pi}{4} + i\sin(-\frac{3\pi}{4}))]$.

Example 0.5. Find $\arg z$ for the complex number $z = 1 + i\tan\frac{3\pi}{5}$.

Let $1 + i\tan\frac{3\pi}{5} = r(\cos\theta + isin\theta)$. Then $r\cos\theta = 1$ and $r\sin\theta = \tan\frac{3\pi}{5}$.

So $r^2 = \sec^2\frac{3\pi}{5} \implies r = -\sec\frac{3\pi}{5}$, as $\sec\frac{3\pi}{5} < 0$.

Therefore $\cos\theta = -\cos\frac{3\pi}{5}$ and $\sin\theta = -\sin\frac{3\pi}{5}$. These gives $\theta = \pi + \frac{3\pi}{5}$. But θ does not give the principal value as $\theta > \pi$. Therefore $\arg z = \theta - 2\pi = -\frac{2\pi}{5}$.

DE MOIVER'S THEOREM

Theorem 0.6. *For all integral values of* n*, the value of* $(\cos\theta + i\sin\theta)^n$ *is* $(\cos n\theta + i\sin n\theta)$ *and for all fractional values of* n*, one of the values of* $(\cos\theta + i\sin\theta)^n$ *is* $(\cos n\theta + i\sin n\theta)$.

Proof:Case 1: When n is a positive integer.

As $(\cos\theta + i\sin\theta)^1 = (\cos\theta + i\sin\theta)$, the theorem holds for $n = 1$.

Let us assume that the theorem is true for $n = m$, where m is a positive integer. So $(\cos\theta + i\sin\theta)^m = (\cos m\theta + i\sin m\theta)$.

Now for $(\cos\theta + i\sin\theta)^{m+1} = (\cos\theta + i\sin\theta)^m(\cos\theta + i\sin\theta) = (\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta) = (\cos m\theta\cos\theta - \sin m\theta\sin\theta) + i(\cos m\theta\sin\theta + \sin m\theta\cos\theta) = \cos(m+1)\theta + i\sin(m+1)\theta$

This shows that the theorem is true for $n = m + 1$. Therefore by the principal of mathematical induction, the theorem is holds for all positive integer n.

Case2: Let n be a negative integer. Again let $n = -m$ where m is a positive integer. Then $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \frac{1}{(\cos\theta+i\sin\theta)^m} = \frac{1}{(\cos m\theta+i\sin m\theta)} = \frac{(\cos m\theta-i\sin m\theta)}{(\cos m\theta-i\sin m\theta)(\cos m\theta+i\sin m\theta)} = \cos m\theta - i\sin m\theta = \cos(-m)\theta + i\sin(-m)\theta = \cos n\theta + i\sin n\theta$

Case3: Let n be a positive fraction. Let $n = \frac{p}{q}$, where p and q are positive integers. Then $(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q})^q = (\cos(q\frac{\theta}{q}) + i\sin(q\frac{\theta}{q})) = (\cos\theta + i\sin\theta)$.

Extracting the qth root of both sides we get that $(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q})$ is one of the values of $(\cos\theta+i\sin\theta)^{\frac{1}{q}}$. Raising each of the quantities to the pth power, we see that $\left(\cos\frac{\theta}{q}+i\sin\frac{\theta}{q}\right)^p$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$.

But $\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p = (\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta)$. Replacing $\frac{p}{q}$ by n, we see that $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

If n be a negative fraction, the theorem can be proved similarly. Hence the theorem is completely proved when n is an integer or a fraction.

Application Of De Moiver's Theorem

There are many application of De Moiver's theorem. We will now discuss two applications of this theorem.

- Determination of all values of $Z^{1/n}$, where n is a positive integer.
- Expansion of cosnθ and sinnθ when n is a positive integer and θ is real.

Determination of all values of $z^{\frac{1}{n}}$ where n is a positive integer.

Let $z = r(\cos\theta + i\sin\theta)$ where $r > 0$, $-\pi < 0 \leq \pi$. By previous theorem we can say that one value of $z^{\frac{1}{n}}$ is $r^{\frac{1}{n}}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})$.We now find out the other values of $z^{\frac{1}{n}}$.

Now $z = r(\cos\theta + i\sin\theta) = r\{\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)\}$ for all integral values of k as we know that the expression $(\cos\theta + i\sin\theta)$ remain unaltered if we put $(\theta + 2k\pi)$ for θ.

Therefore by De Moiver's theorem, the nth root of z are

$r^{\frac{1}{n}}\Big\{\cos(2k\pi+\theta)+i\sin(2k\pi+\theta)\Big\}^{\frac{1}{n}} = r^{\frac{1}{n}}\Big\{\cos\frac{(2k\pi+\theta)}{n}+i\sin\frac{(2k\pi+\theta)}{n}\Big\}$ where $k = 0, 1, \ldots, (n-1)$

If values greater than $n-1$, i.e $n, n+1, n+2, \ldots$ be giving to k, then we would get the same quantities already obtaied by putting $k = 0, 1, 2, \ldots, n-1$ repeated over and over again.Also no two quantities, as obtained by putting $k = 0, 1, 2, \ldots, n-1$ are the same. Thus $z^{\frac{1}{n}}$ has n distinct values $r^{\frac{1}{n}}\Big\{\cos\frac{2k\pi+\theta}{n} + i\sin\frac{2k\pi+\theta}{n}\Big\}$, where $k = 0, 1, \ldots, n-1$

Expansion of $\cos n\theta$ and $\sin n\theta$ when n is a positive integer and θ is real.

By De, Moiver's theoerem we have, $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$, when n is a positive integer. As $\cos\theta \leq 1$ and $\sin\theta \leq 1$ for all values of θ, we can apply Binomial theorem. Therefore we have

$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n = \cos^n\theta +^n C_1 \cos^{n-1}\theta(i\sin\theta) + \cdots + (i\sin\theta)^n$

$=\{\cos^n\theta -^n C_2 \cos^{n-2}\theta\sin^2\theta + \dots\} + i\{^nC_1\cos^{n-1}\theta\sin\theta -^n C_3\cos^{n-3}\theta\sin^3\theta + \dots\}$

Equating the real and imaginary parts from both sides, we have

$\cos n\theta = \cos^n\theta -^n C_2\cos^{n-2}\theta\sin^2\theta + \dots$ and

$\sin n\theta =^n C_1\cos^{n-1}\theta\sin\theta -^n C_3\cos^{n-3}\theta\sin^3\theta + \dots$

If n be odd the last term in the expansionof $\cos n\theta$ is $(-1)^{\frac{(n-1)}{2}}\,{}^nC_{n-1}\cos\theta\sin^{n-1}\theta$ and that of $\sin n\theta$ is $(-1)^{\frac{(n-1)}{2}}\sin^n\theta$.

If n be even, the last term in the expransion of $\cos n\theta$ is $(-1)^{\frac{n}{2}}\sin^n\theta$ and that of $\sin n\theta$ is $(-1)^{\frac{(n-1)}{2}}\,{}^nC_{n-1}\cos\theta\sin^{n-1}\theta$.

The series for $\cos n\theta$ and $\sin n\theta$ are thus alternating.