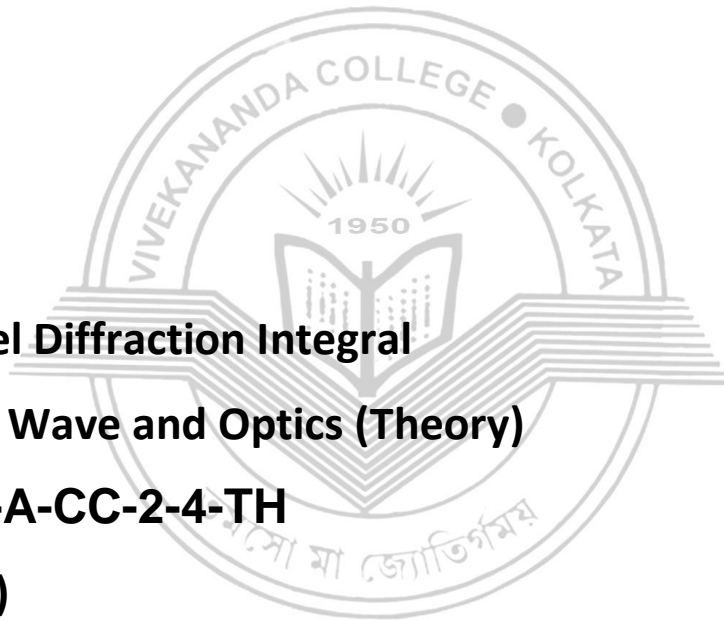


**VIVEKANANDA COLLEGE
THAKURPUKUR
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NAAC ACCREDITED 'A' GRADE



Topic: Fresnel Diffraction Integral

Course Title: Wave and Optics (Theory)

Paper: PHS-A-CC-2-4-TH

Unit: 2.2.1(8)

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Name of the Department: PHYSICS

Fresnel diffraction integral process is more generalised analysis of the diffraction phenomena of a plane wave by different types of aperture. Let us considered a plane wave (of amplitude A) incident normally on an aperture of arbitrary shape as shown in Fig.- 1 on ξ - η plane. We have to find out the resultant disturbance at point P on x-y plane, which is placed at a distance z from the ξ - η plane. According to Huygens-Fresnel principle we can say the incident wave on x-y plane is a spherical wave.

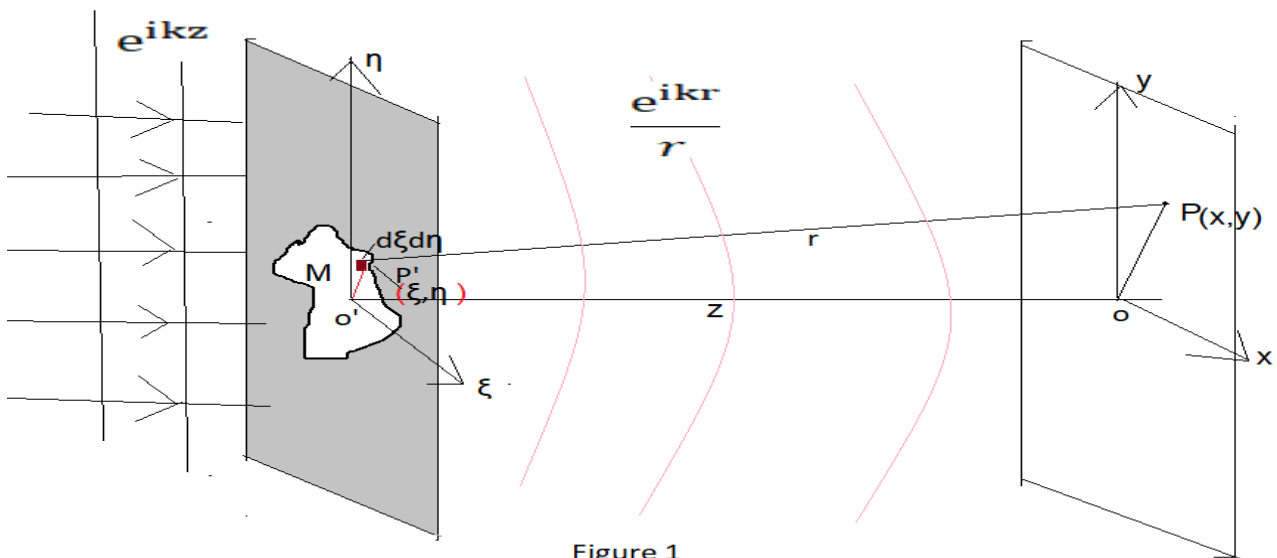


Figure 1

Now we have to introduce a term called transmission function i.e.

$$U(\xi, \eta) = 1 \text{ inside the aperture.}$$

$$= 0 \text{ outside the aperture.}$$

The disturbance at P due to the secondary wavefront from the elementary area $d\xi d\eta$ is proportional to $A \cdot \frac{e^{ikr}}{r} d\xi d\eta$ where $r = MP$.

So the total disturbance at P due to the secondary wave from the whole aperture is

$$u(x, y) = u(p) \propto \iint_{-\infty}^{+\infty} A \frac{e^{ikr}}{r} d\xi d\eta \text{ -----(1)}$$

$$u(p) = C \iint_{-\infty}^{+\infty} A \frac{e^{ikr}}{r} d\xi d\eta \text{ -----(2)}$$

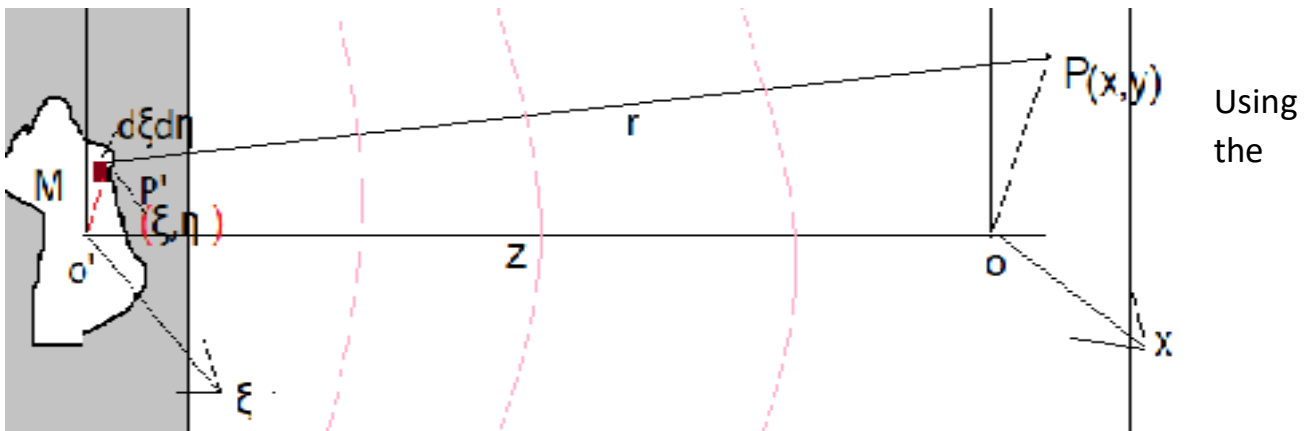
First we have to find out the value of proportionality constant C by using some boundary conditions.

The boundary condition is if there is no opaque space i.e. the whole ξ - η plane behaves as aperture then the incident wave at P is also a plane wave and that is e^{ikr}

Then it can be written as

$$e^{ikr} = C \iint_{-\infty}^{+\infty} \frac{e^{ikr}}{r} d\xi d\eta \text{ -----(3)}$$

'A' is omitted as the whole space is aperture.



Using the

coordinate geometry we can say $r = [(x - \xi)^2 + (y - \eta)^2 + z^2]^{1/2}$

$$r = z[1 + \left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2]^{1/2} \text{ -----(4)}$$

Let us consider $|x - \xi| \ll z$ and $|y - \eta| \ll z$

Then using binomial expansion it can be written as

$$r = z[1 + \frac{1}{2}\left(\frac{x-\xi}{z}\right)^2 + \frac{1}{2}\left(\frac{y-\eta}{z}\right)^2] \text{ -----(5)}$$

Putting the value of r in exponential part of spherical waveform and $r = z$ in linear part we get

$$e^{ikz} = C \iint_{-\infty}^{+\infty} \frac{e^{ikz[1 + \frac{1}{2}\left(\frac{x-\xi}{z}\right)^2 + \frac{1}{2}\left(\frac{y-\eta}{z}\right)^2]}}{z} d\xi d\eta$$

$$e^{ikz} = C \frac{e^{ikz}}{z} \iint_{-\infty}^{+\infty} \exp\left[\frac{ik}{2z}(x - \xi)^2 + \frac{ik}{2z}(y - \eta)^2\right] d\xi d\eta$$

$$e^{ikz} = C \frac{e^{ikz}}{z} \int_{-\infty}^{\infty} e^{\frac{ik}{2z}(x-\xi)^2} d\xi \int_{-\infty}^{+\infty} e^{\frac{ik}{2z}(y-\eta)^2} d\eta \text{ -----(6)}$$

$$e^{ikz} = C \frac{e^{ikz}}{z} . I1 . I2 \text{ -----(7)}$$

$$\text{Where } I1 = \int_{-\infty}^{\infty} e^{\frac{ik}{2z}(x-\xi)^2} d\xi \text{ and } I2 = \int_{-\infty}^{+\infty} e^{\frac{ik}{2z}(y-\eta)^2} d\eta \text{ -----(8)}$$

$$I_1 = \int_{-\infty}^{\infty} e^{\frac{ik}{2z}(x-\xi)^2} d\xi = \int_{-\infty}^{\infty} e^{\frac{ik}{2z}(x^2 + \xi^2 - 2x\xi)} d\xi = \int_{-\infty}^{\infty} e^{\frac{ik}{2z}x^2} e^{\frac{ik}{2z}\xi^2} e^{-\frac{ik}{z}x\xi} d\xi \quad \text{---(9)}$$

We know from β - γ function $\int_{-\infty}^{\infty} e^{-\alpha x^2} e^{\beta x} dx = \sqrt{\frac{\pi}{\alpha}} \times e^{\frac{\beta^2}{4\alpha}}$ -----(10)

Comparing equation (9) and (10) we have $\alpha = -\frac{ik}{2z}$ and $\beta = -\frac{ikx}{z}$

So $I_1 = e^{\frac{ik}{2z}x^2} \sqrt{\frac{\pi}{-\frac{ik}{2z}}} \times e^{\frac{(-\frac{ikx}{z})^2}{4(-\frac{ik}{2z})}} = e^{\frac{ikx^2}{2z}} \sqrt{\frac{2\pi z}{-ik}} \times e^{\frac{-ikx^2}{2z}} = \sqrt{i\left(\frac{2\pi}{k}\right)z} = \sqrt{i\lambda z}$ -----(11)

We know $\lambda = \left(\frac{2\pi}{k}\right)$.

Similarly $I_2 = \sqrt{i\lambda z}$; putting I_1 and I_2 in equation 7 we get

$$e^{ikz} = C \frac{e^{ikz}}{z} \sqrt{i\lambda z} \sqrt{i\lambda z}$$

$$e^{ikz} = C \frac{e^{ikz}}{z} (i\lambda z) \Rightarrow C = \frac{1}{i\lambda} = -\frac{i}{\lambda} \quad \text{---(12)}$$

So we have the resultant disturbance at P due to arbitrary aperture

$$u(x,y) = u(p) = -\frac{iA}{\lambda} \iint_{-\infty}^{+\infty} \frac{e^{ikr}}{r} d\xi d\eta \quad \text{---(13)}$$

Diffraction of a plane wave normally incident on a circular aperture:

We assume a plane wave incident normally on a circular aperture of radius a as shown in Fig. 2. The z -axis is normal to the plane of the aperture and the screen SS' is assumed to be normal to the z -axis. It is obvious from the symmetry of the problem

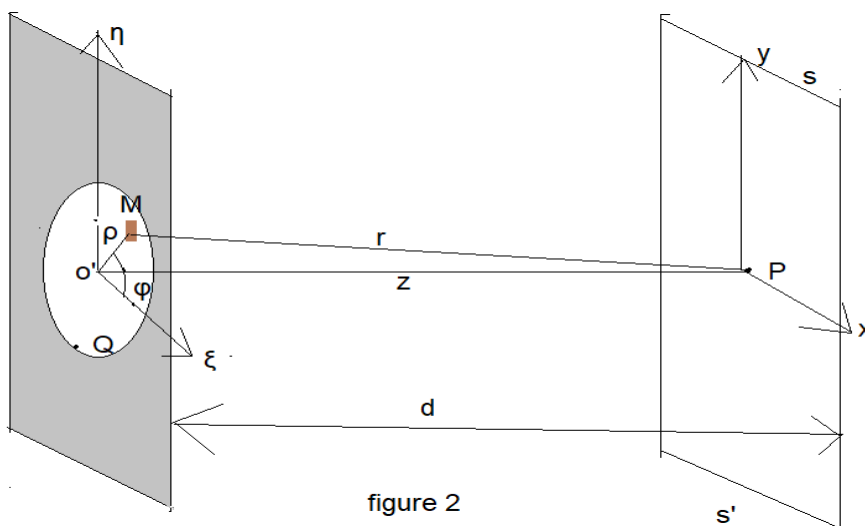


figure 2

that we will obtain circular fringes on the screen SS' ; however, it is very difficult to calculate the actual intensity variation on the screen. Therefore, for the sake of mathematical simplicity, we will calculate the variation of intensity

only along the z-axis. Obviously, it will be more convenient to use the circular system of coordinates. In this system, the coordinates of an arbitrary point M on the aperture will be (ρ, ϕ) where ρ is the distance of the point M from the centre O and ϕ is the angle that OM makes with the ξ -axis [see Fig. 2] and a small element area dS surrounding the point M will be $\rho d\rho d\phi$. Thus

$$u(p) = -\frac{iA}{\lambda} \iint_{-\infty}^{+\infty} \frac{e^{ikr}}{r} d\xi d\eta = u(p) = -\frac{iA}{\lambda} \int_0^{2\pi} d\phi \int_0^a \frac{e^{ikr}}{r} \rho d\rho \text{ -----(14)}$$

Now $\rho^2 + d^2 = r^2$ so $\rho d\rho = r dr$

Putting in equation (14)

$$u(p) = -\frac{iA}{\lambda} \int_0^{2\pi} d\phi \int_d^{\sqrt{a^2+d^2}} \frac{e^{ikr}}{r} r dr$$

$$u(p) = -\frac{iA}{\lambda} \times 2\pi \times \frac{e^{ikr\sqrt{a^2+d^2}}}{ik d} = -\frac{iA}{\lambda} \times \frac{2\pi}{ik} \times [e^{ik\sqrt{a^2+d^2}} - e^{ikd}]$$

Putting $k = 2\pi/\lambda$

$$u(p) \approx Ae^{ikd} (1 - e^{ip\pi}) \text{ -----(15)}$$

Where $p\pi = k[\sqrt{a^2 + d^2} - d]$ and from the figure-2 we have a point Q on the periphery of the aperture.

$$\text{So } QP = \sqrt{a^2 + d^2}, OP = d. \text{ or } QP - OP = p\pi/k = p\frac{\lambda}{2} \text{ -----(16)}$$

From the equation (15)

$$u(p) \approx 2iAe^{ikd} e^{ip\pi/2} \left(\frac{e^{-ip\pi/2} - e^{ip\pi/2}}{2i} \right)$$

$$u(p) \approx -2iAe^{ikd} e^{\frac{ip\pi}{2}} \sin\left(\frac{p\pi}{2}\right) \text{ -----(17)}$$

$$\text{Intensity } I \propto u(p) \cdot u(p)^* \propto [-2iAe^{ikd} e^{\frac{ip\pi}{2}} \sin\left(\frac{p\pi}{2}\right)] [2iA^* e^{-ikd} e^{-\frac{ip\pi}{2}} \sin\left(\frac{p\pi}{2}\right)]$$

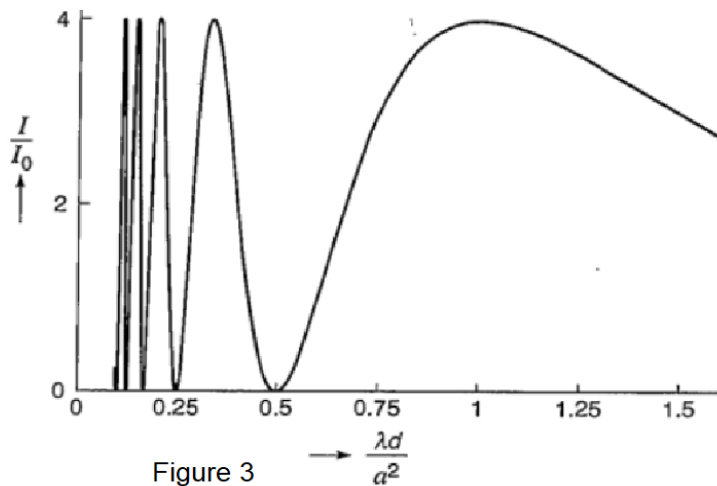
$$I = 4I_0 \sin^2(p\pi/2) \text{ -----(18)}$$

Where I_0 is the intensity associated with the incident plane wave. Equation (18) tells us that the intensity is zero or maximum when p is an even or odd integer, i.e., when $QP - OP$ is an even or odd multiple of $\lambda/2$. This can be understood physically by using the concept of Fresnel half-period zones discussed in earlier note. Thus, if the aperture contains an even number of half-period zones, the intensity at the point P

will be negligibly small and conversely, if the circular aperture contains an odd number of zones, the intensity at the point P will be maximum. Now, when $d \ll a$ (as is usually the case)

$$P = \frac{k}{\pi} \left[d \left(1 + \frac{a^2}{2d^2} \right) - d \right] = \frac{a^2}{\lambda d} \text{-----(19)}$$

Which is known as the Fresnel number of the aperture.



In Fig. 3 we have plotted the corresponding intensity variation as a function of the dimensionless parameter $\frac{\lambda d}{a^2}$.

The figure shows that when the (circular) aperture contains an even number of half-period zones, the intensity at the point P will be zero and when the

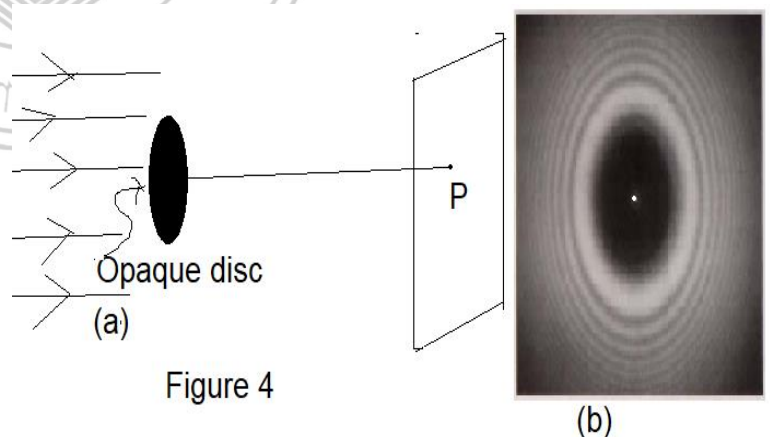
aperture contains an odd number of zones, the intensity at the point P will be maximum.

Diffraction by an Opaque Disc-The Poisson Spot :

If instead of the circular aperture we have a circular disc [see Fig. 4(a)] and if the disc obstructs the first p half-period zones then the field at the point P would be

$$U(p) = u_{p+1} - u_{p+2} + \dots \approx u_{p+1}/2 \text{---(20)}$$

Fig. 4 (a) When a plane wave is incident normally on an opaque disc, a bright spot is always formed on an axial point. This spot is known as the Poisson spot, (b) The Poisson spot at the center of the shadow of a one cent coin;



Thus, we should always obtain a bright spot on the axis behind a circular disc; (the more rigorous theory also predicts the same result). This is called the 'Poisson spot'.

Using equation (20) once again we will assume that the observation point lie on the axis of the disc. Equation (2) tells us that in order to calculate the field we have to

carry out an integration over the open region of the aperture. Obviously, if $u_1(P)$ and $u_2(P)$ respectively represent the fields at the point P due to a circular aperture and an opaque disc (of the same radius), then

$$u_1(P) + u_2(P) = u_0(P) \text{ -----(21)}$$

Where $u_0(P)$ represents the field in the absence of any aperture; Eq. (21) is known as the Babinet's principle. Thus,

$$u_2(P) = u_0(P) - u_1(P);$$

using (15) where Ae^{ikd} is the disturbance at P without any opaque space = $u_0(P)$
 $= u_0(P) - u_0(P) (1 - e^{ip\pi})$

$$u_2(P) = u_0(P) e^{ip\pi} \text{ -----(22)}$$

Thus the intensity at P due to an opaque disc is

$$I_2(P) = |u_2(P)|^2 = I_0(P) \text{ -----(23)}$$

Which gives us the remarkable result that the intensity at a point on the axis of an opaque disc is equal to the intensity at the point in the absence of the disc! This is the Poisson spot.

