VIVEKANANDA COLLEGE THAKURPUKUR KOLKATA-700063

NAAC ACCREDITED ‘A’ GRADE

Topic: Phase Velocity and Group Velocity

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Phase velocity and Group velocity.

A plane progressive harmonic wave propagating in the positive x-direction can be expressed as

$$y = A \sin \frac{2\pi}{\lambda}(vt - x).$$

$$\text{or, } y = A \sin(\omega t - Kx)$$

ω = angular frequency
K = propagation const.
$\because K = \frac{2\pi}{\lambda}$ and $\omega = \frac{2\pi}{T}$
and $\frac{v}{\lambda} = \frac{1}{T}$.

The wave moves with velocity 'v' $= \cancel{dx} \; \frac{\omega}{K}$

'v' is also referred as the phase velocity or wave velocity which depends on elastic property and density of the medium.

If two or more plane progressive harmonic waves of the same amplitude but different frequencies superimpose, a group of wave is formed. The amplitude of the group changes with distance and the velocity with which the maximum of the wave group travels is referred as the group velocity. The energy is transmitted with the group velocity.

For small amplitude acoustic waves moving in an isotropic homogeneous elastic medium, the wave velocity does not depends on frequency.

The velocities of the component waves being equal, the wave group does not change its form as it advances; thus the group velocity is same as the phase velocity.

If the medium is dispersive i.e. if the phase velocity depends on the frequency, then the group velocity will be different from the phase velocity.

Considering a wave group is formed by the superimposition of two waves (simplest case) of equal amplitude 'A' and slightly different angular frequencies 'ω' and 'ω+dω', travelling with the propagation constants 'K' and 'K+dK' respectively.

∴ The resultant displacement

$$y = A\sin(\omega t - Kx) + A\sin[(\omega + d\omega)t - (K + dK)x]$$

upon simplification

$$\text{or, } y = 2A\cos\left(\frac{t\,d\omega - x\,dk}{2}\right)\sin\left[\left(\omega + \frac{d\omega}{2}\right)t - \left(K + \frac{dk}{2}\right)x\right].$$

$$\left[\text{using } \sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{B-A}{2}\right]$$

The slowly varying cosine term is the amplitude of the resultant wave.

The angular frequency and propagation constants of the resultant wave are actually the mean values of the same of ~~componint~~ component waves.

The phase velocity of composite wave is

$$v = \frac{\omega + \frac{d\omega}{2}}{K + \frac{dK}{2}} \simeq \frac{\omega}{K}$$ [nearly equal to the phase velocity of component waves]

The amplitude of the resultant wave advances with group velocity v_g.

The group velocity $v_g = \frac{d\omega}{dK}$

From phase velocity $v = \frac{\omega}{K}$

$$\therefore \omega = vK.$$

$$\text{or, } d\omega = v\,dK + K\,dv.$$

$$\therefore v_g = \frac{d\omega}{dK} = v + K\frac{dv}{dK}$$

$$\text{or, } v_g = v + K\frac{dv}{d\lambda}\cdot\frac{d\lambda}{dK}.$$

$$\text{or, } v_g = v - \frac{2\pi}{K}\frac{dv}{d\lambda}.$$

$$\left[\because K = \frac{2\pi}{\lambda}.\quad \frac{d\lambda}{dK} = -\frac{2\pi}{K^2}\right]$$

$$\boxed{v_g = v - \lambda\frac{dv}{d\lambda}}$$

The above expression is the relation between phase velocity and group velocity.

In a non-dispersive medium, the phase velocity is independent of the frequency (or wavelength), i.e. $\frac{dv}{d\lambda} = 0$ and so, $v_g = v$

$\therefore$ phase velocity = group velocity.

But in dispersive medium, where 'v' (phase velocity) increases with increasing λ, $\frac{dv}{d\lambda}$ is positive and so $v_g < v$,

$\therefore$ group velocity is less than phase velocity.

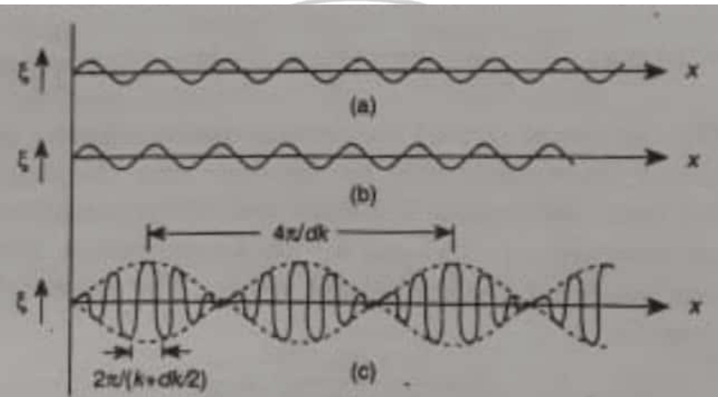

Fig. 5.2 The component waves are given in (a) and (b); the resultant wave in (c)