VIVEKANANDA COLLEGE
THAKURPUKUR
KOLKATA-700063

NAAC ACCREDITED ‘A’ GRADE

Topic: Fourier transform- Class-1

Course Title: Mathematical Physics

Paper:

Unit:

Semester: IV

Name of the Teacher: Dr. Nasrin Banu

Name of the Department: Physics

Fourier Transform

Integral transform

If two functions $f(x)$ and $g(x)$ are related to each other by the expression

$$\boxed{g(x) = \int_a^b f(t)\, K(x,t)\, dt}$$

$g(x)$ is called integral transform (I.T.) of $f(t)$ and $K(x,t)$ is called the Kernel of the transformation.

$f(t)$ is called inverse integral transform of $g(x)$.

Depending on a, b and $K(x,t)$ there are different types of integral transforms.

E.g.

① Fourier transform (F.T.) : $K(x,t) = e^{ixt}$; $a = -\infty, b = \infty$

$$\boxed{g(x) = \text{F.T. of } f(t) = \int_{-\infty}^{\infty} f(t)\, e^{ixt}\, dt}$$

② Laplace's transform : $K(x,t) = e^{-xt}$; $a = 0, b = \infty$

$$\boxed{g(x) = \text{L.T. of } f(t) = \int_0^{\infty} f(t)\, e^{-xt}\, dt}$$

③ Hankel transform : $K(x,t) = t\, J_n(xt)$; $a = 0, b = \infty$

$$\boxed{g(x) = \text{H.T. of } f(t) = \int_0^{\infty} f(t)\, t\, J_n(xt)\, dt}$$

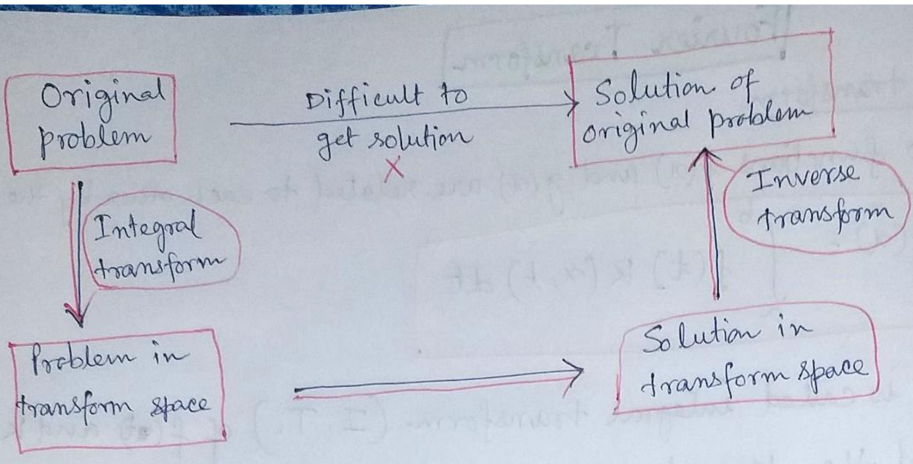
$J_n(xt)$ is Bessel function

④ Mellin transform : $K(x,t) = t^{x-1}$; $a = 0, b = \infty$

$$\boxed{g(x) = \text{M.T. of } f(t) = \int_0^{\infty} f(t)\, t^{x-1}\, dt}$$

Importance of Integral Transform

Some times, the solution of any problem becomes very difficult in direct space or physical space. In such problems the solution may be easier to find in transformed space. In such cases the following steps are used to solve the problem. For example, under Laplace transform differential equations become algebric equation and we know that solving algebric equation is easier than solving differential equation.



Fourier Transform

Fourier transform (F.T.) of function $f(t)$ is $g(x)$ given by

$$g(x) = \int_{-\infty}^{\infty} f(t)\, e^{ixt}\, dt$$

Example of use of Fourier transform:

① Signal processing: $f(t)$ represents the time behaviour of a signal. The corresponding frequency distribution is the fourier transform of $f(t)$ which is $g(\omega)$.

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

The prefactor $\frac{1}{\sqrt{2\pi}}$ is used here to make the transform and inverse transform formulas symmetrical.

② They are used in evaluating integrals.

③ Alternative formulations of quantum mechanics.

Fourier cosine and fourier sine Transform

F.T. of function $f(t)$ $g(\omega)$ is given by

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\cos\omega t + i\sin\omega t\right] dt$$

If $f(t)$ is an even function then $f(t)\sin\omega t$ is an odd function and $\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt = 0$

$\therefore$ for even functions $f(t)$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\cos\omega t\, dt$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty f(t) \cos\omega t \, dt \qquad \left[\int_{-\infty}^{\infty} (\text{even func}^n) \Rightarrow 2\int_0^\infty \dots \right]$$

or, $$\boxed{g(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos\omega t \, dt}$$ for even functions $f(t)$. — ①

Similarly for odd function $f(t)$

$$\boxed{g(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin\omega t \, dt}$$ for odd $f(t)$ — ②

The formula ① defines fourier cosine transform and formula ② represents fourier sine transform.

The kernel in both F.T. ① & ② are real.

These two transformations are used in the studies of wave motion.

* The electron distribution in atom may be obtained from Fourier transform of the amplitude of the scattered X-rays.

Fourier integral theorem

Under certain conditions function $f(x)$ can be expressed as

$$\boxed{f(x) = \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^{\infty} f(t) \cos u(t-x) \, dt \, du}$$

The conditions are

① $f(x)$ is single valued function except finite points $(-l, l)$.

② $f(x)$ is periodic with period $2l$.

Fourier series of the function $f(x)$ in $(-l, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where, a_0, a_n and b_n are given by

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(t)\,dt \ ; \ a_n = \frac{1}{l}\int_{-l}^{l} f(t) \cos\frac{n\pi t}{l}\,dt \ ; \ b_n = \frac{1}{l}\int_{-l}^{l} f(t) \sin\frac{n\pi t}{l}\,dt$$

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \sum_{n=1}^{\infty} \frac{1}{l}\int_{-l}^{l} f(t) \cos\frac{n\pi t}{l} \cos\frac{n\pi x}{l}\,dt$$

$$+ \sum_{n=1}^{\infty} \frac{1}{l}\int_{-l}^{l} f(t) \sin\frac{n\pi t}{l} \cdot \sin\frac{n\pi x}{l}\,dt$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \sum_{n=1}^{\infty} \frac{1}{l}\int_{-l}^{l} f(t)\left[\cos\frac{n\pi t}{l}\cos\frac{n\pi x}{l} + \sin\frac{n\pi t}{l}\sin\frac{n\pi x}{l}\right]dt$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \sum_{n=1}^{\infty} \frac{1}{l}\int_{-l}^{l} f(t) \cos\frac{n\pi}{l}(t-x)\,dt$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\left[1 + 2\sum_{n=1}^{\infty} \cos\frac{n\pi}{l}(t-x)\right]dt$$

$$1 + 2\sum_{n=1}^{\infty} \cos\frac{n\pi}{l}(t-x) = \sum_{-\infty}^{\infty} \cos\frac{n\pi}{l}(t-x)$$

$$\therefore f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\sum_{-\infty}^{\infty}\cos\frac{n\pi}{l}(t-x)\,dt$$

$$= \frac{1}{2\pi}\int_{-l}^{l} f(t)\frac{\pi}{l}\sum_{-\infty}^{\infty}\cos\frac{n\pi}{l}(t-x)\,dt$$

Let $\frac{n\pi}{l} = u$, $\frac{\pi}{l} = du$; l increases indefinitely

Then $\underset{l\to\infty}{Lt}\ \frac{\pi}{l}\sum_{-\infty}^{\infty}\cos\frac{n\pi}{l}(t-x) = \int_{-\infty}^{\infty}\cos u(t-x)\,du = 2\int_0^{\infty}\cos u(t-x)\,du$ as cosine func^n is even func^n

$$\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left\{2\int_0^{\infty}\cos u(t-x)\,du\right\}dt$$

$$\boxed{f(x) = \frac{1}{\pi}\int_{t=-\infty}^{\infty}\int_{u=0}^{\infty} f(t)\cos u(t-x)\,du\,dt}$$

Examples of Fourier transform

1. $f(t) = e^{-\alpha|t|}$ with $\alpha > 0$.

$$\text{F.T. of } f(t) = g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{i\omega t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha|t|}\cdot e^{i\omega t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{\alpha t}\cdot e^{i\omega t}\,dt + \int_0^{\infty} e^{-\alpha t}\,e^{i\omega t}\,dt\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{(\alpha+i\omega)t}\,dt + \int_0^{\infty} e^{-(\alpha-i\omega)t}\,dt\right]$$

$$\left[\frac{1}{-(\alpha-i\omega)}\left\{e^{-(\alpha-i\omega)\infty} - e^{0}\right\} = \frac{1}{\alpha-i\omega}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\alpha+i\omega} + \frac{1}{\alpha-i\omega}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{\alpha-i\omega+\alpha+i\omega}{(\alpha+i\omega)(\alpha-i\omega)}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{2\alpha}{\alpha^2+\omega^2}$$

This is real. So, Fourier transform of even function is real.

If $f(t)$ is more localized $g(\omega)$ will be less localized.

Fourier transform of odd functions are imaginary.

Functions neither even nor odd have Fourier transform in complex form.

2. $f(t) = \delta(t)$ (Dirac delta function)

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t)\, e^{i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 1$$

$$= \frac{1}{\sqrt{2\pi}}$$

In time domain $f(t)$ is the utmost localized function. But we see $g(\omega)$ is completely delocalized in frequency domain. It has same value $\frac{1}{\sqrt{2\pi}}$ for all ω.

3. $f(t) = \frac{2\alpha}{\sqrt{2\pi}(\alpha^2+t^2)}$. To evaluate this F.T. we need to know contour integration. Using this method we can calculate easily. Will be shown later.

Properties of Fourier transform

Let $g(\vec{k})$ be the Fourier transform of $f(\vec{r})$. Then,

1. $\boxed{\text{F.T. of } f(\vec{r}-\vec{R}) = e^{i\vec{k}\cdot\vec{R}}\, g(\vec{k})}$ ⇒ Translation property
2. $\boxed{\text{F.T. of } f(\alpha\vec{r}) = \frac{1}{\alpha^3}\, g(\alpha^{-1}\vec{k})}$ ⇒ change of scale
3. $\boxed{\text{F.T. of } f(-\vec{r}) = g(-\vec{k})}$ ⇒ change of sign
4. $\boxed{\text{F.T. of } f^*(-\vec{r}) = g^*(\vec{k})}$ ⇒ Complex conjugation
5. $\boxed{\text{F.T. of } \vec{\nabla} f(\vec{r}) = -i\vec{k}\, g(\vec{k})}$ ⇒ gradient
6. $\boxed{\text{F.T. of } \vec{\nabla}^2 f(\vec{r}) = -k^2\, g(\vec{k})}$ ⇒ Laplacian

All the Fourier transforms here are written in 3D form to get the general form. In 1D the properties 5 and 6 are given below

(5) F.T. of $f'(t) = -i\omega\, g(\omega)$

(6) F.T. of $\frac{d^n f(t)}{dt^n} = (-i\omega)^n g(\omega)$

Fourier transform in 3D space

F.T. of function $f(\vec{r})$ in 3D space is $g(\vec{k})$ and is given by

$$\boxed{g(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} f(\vec{r})\, e^{i\vec{k}\cdot\vec{r}}\, dr}$$

(6)

Inverse Fourier Transform

If $f(t)$ is the function in original space and $g(\omega)$ is the Fourier Transform of $f(t)$ such that

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

then, $f(t)$ is inverse fourier transform of $g(\omega)$ and is given by

$$\boxed{f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega)\, e^{-i\omega t}\, d\omega}$$

The difference in the formulas of Fourier transform and inverse Fourier transform is only the sign of exponential which is complex.

Inverse Fourier sine and cosine transform

Fourier cosine transform of even functions is given by

$$g(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \cos\omega t\, dt$$

and inverse Fourier cosine transform is -

$$f(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} g(\omega) \cos\omega t\, d\omega$$

Similarly ~~f~~ inverse Fourier ~~co~~sine transform of odd function is

$$f(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} g(\omega) \sin\omega t\, d\omega$$

Fourier Integral

Deltafunction is represented by

$$\delta_n(t) = \frac{1}{2\pi} \int_{-n}^{n} e^{i\omega t}\, d\omega$$

For ~~Any~~ function $f(x)$

$$f(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)\, \delta_n(t-x)\, dt$$

$$= \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \left[\int_{-n}^{n} e^{i\omega(t-x)}\, d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \lim_{n\to\infty} \int_{-n}^{n} e^{i\omega(t-x)}\, d\omega\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} e^{i\omega(t-x)}\, d\omega\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} f(t)\, e^{i\omega(t-x)}\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x} d\omega \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \quad \text{---} (1)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x} d\omega \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

or, $$f(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} g(\omega)\, e^{-i\omega x}\, d\omega$$

$$\Rightarrow \boxed{f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)\, e^{-i\omega t}\, d\omega}$$ [putting $x = t$]

Rewritting eq^n (1)

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x} d\omega \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt}$$

is the Fourier Integral, is integral ~~ka~~ representation of $f(x)$