VIVEKANANDA COLLEGE THAKURPUKUR KOLKATA-700063

NAAC ACCREDITED ‘A’ GRADE

Topic: Energy of a Vibrating String

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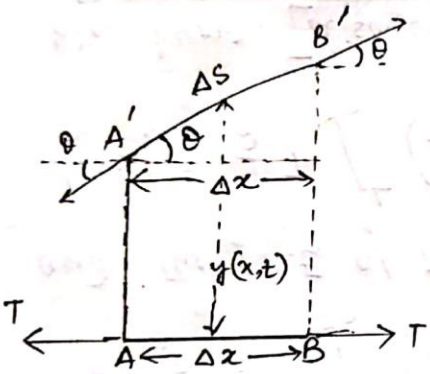
Energy of a Vibrating String

A vibrating string possesses both kinetic and potential energy. The kinetic energy is due to the velocity of the element at that instant and the potential energy is the work done by the tension 'T' in extending the element AB of length Δx to ΔS ($A'B'$) due to a displacement 'y' from the equillibrium position.



$$\frac{\Delta x}{\Delta S} = \cos\theta \quad [\text{from fig}]$$

$$\therefore \frac{(\Delta S)^2}{(\Delta x)^2} = \sec^2\theta = 1 + \tan^2\theta$$

$$\therefore \frac{(\Delta S)^2}{(\Delta x)^2} = 1 + \left(\frac{\partial y}{\partial x}\right)^2$$

$$\therefore \Delta S = \left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{1/2} \Delta x$$

$$\text{or, } \Delta S = \left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right] \Delta x$$

neglecting higher order terms

Therefore, the change in the length of the element is $\boxed{(\Delta S - \Delta x) = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 \Delta x}$

The instantaneous kinetic energy of ΔS

$= \frac{1}{2}\mu \Delta S \left(\frac{\partial y}{\partial t}\right)^2$ $\qquad \mu$ = linear mass density

The total K.E of the string of length 'L',

$$K.E = \frac{1}{2}\mu \int_0^L \left(\frac{\partial y}{\partial t}\right)^2 dx$$

[$\Delta S = \Delta x$ since y and its derivatives are assumed to be small.]

The instantaneous potential energy of the element is the work done due to extension.

$$\therefore P.E = \int_0^L T(\Delta s - \Delta x)$$

$$P.E = \frac{1}{2} T \int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx$$

The total energy of a vibrating string is the sum of the energies associated with all the normal modes of the string.

Thus the K.E of the 'n^{th}' mode is

$$(K.E)_n = \frac{1}{2} \mu \int_0^L \left(\frac{\partial y_n}{\partial t}\right)^2 dx .$$

and the P.E of the 'n^{th}' mode is

$$(P.E)_n = \frac{1}{2} T \int_0^L \left(\frac{\partial y_n}{\partial x}\right)^2 dx .$$

We know, [discussed earlier]. $\left[\begin{array}{l} \omega_n = \frac{n\pi v}{l} \\ k_n = \frac{n\pi}{l} \text{ i.e. } \frac{\omega_n}{v} = k_n \end{array}\right]$

$$\boxed{y_n = (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin k_n x .}$$

$$\therefore \left(\frac{\partial y_n}{\partial t}\right) = \omega_n \left(-A_n \sin \omega_n t + B_n \cos \omega_n t\right) \sin k_n x .$$

$$\text{and } \left(\frac{\partial y_n}{\partial x}\right) = k_n \left(A_n \cos \omega_n t + B_n \sin \omega_n t\right) \cos k_n x .$$

$$\therefore (K.E)_n = \frac{1}{2}\mu\omega_n^2(-A_n \sin\omega_n t + B_n \cos\omega_n t)^2 \int_0^L \sin^2(K_n x)\,dx$$

$$\& \ (P.E)_n = \frac{1}{2} T K_n^2 (A_n \cos\omega_n t + B_n \sin\omega_n t)^2 \int_0^L \cos^2(K_n x)\,dx.$$

Here, $k_n = \frac{\omega_n}{v} = \omega_n\sqrt{\frac{\mu}{T}}$ $\qquad \because v = \sqrt{\frac{T}{\mu}}.$

$$\therefore \ TK_n^2 = \mu\omega_n^2$$

$\therefore$ Total energy in 'n'th mode,

$$E_n = (K.E)_n + (P.E)_n$$

$$\text{or, } E_n = \frac{1}{2}\mu\omega_n^2 \cdot \frac{L}{2}\left(A_n^2 \sin^2\omega_n t + B_n^2\cos^2\omega_n t - 2A_nB_n \sin\omega_n t \cos\omega_n t\right)$$

$$+ \frac{1}{2} T K_n^2 \cdot \frac{L}{2}\left(A_n^2 \cos^2\omega_n t + B_n^2 \sin^2\omega_n t + 2A_nB_n \sin\omega_n t \cos\omega_n t\right)$$

$$\text{or, } E_n = \frac{1}{4}\mu L\omega_n^2(A_n^2 + B_n^2).$$

$$\therefore E_n = \frac{1}{4} m\omega_n^2 (A_n^2 + B_n^2) \qquad \because [m = \mu L = \text{mass of string}].$$

$$\therefore \boxed{E_{total} = \frac{1}{4} m \sum_{n=1}^{\infty}\{\omega_n^2(A_n^2 + B_n^2)\}}$$

Energy Transport in Travelling Waves.

Consider a wave travelling on a string in the positive x-direction. The various particles of the string along the direction of propagation are set into vibration in succession. Thus, a transfer of energy from one part of the string to another occurs.

The particle displacements

$$\boxed{y(x,t) = A \sin \frac{2\pi}{\lambda}(vt - x)}$$ where $v = \sqrt{\frac{T}{\mu}}$.

'T' → Tension in the string and.
'μ' → linear density of mass.

Let us consider a small element 'dx' of the string.

The potential energy of this element will be

$$\boxed{dU = \frac{1}{2} T \left(\frac{\partial y}{\partial x}\right)^2 dx.}$$ and

the kinetic energy of the element is given by

$$\boxed{dK = \frac{1}{2} \mu \left(\frac{\partial y}{\partial t}\right)^2 dx.}$$

$\therefore$ $\frac{dU}{dx} = \frac{1}{2} T \left(\frac{\partial y}{\partial x}\right)^2$ ⇒ Potential energy density

& $\frac{dK}{dx} = \frac{1}{2} \mu \left(\frac{\partial y}{\partial t}\right)^2$ ⇒ Kinetic energy density.

$$\therefore \quad \frac{du}{dx} = \frac{1}{2} T A^2 \left(\frac{2\pi}{\lambda}\right)^2 \cos^2\left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

[using the expression of 'y']

$$\text{and} \quad \frac{dk}{dx} = \frac{1}{2}\mu A^2 \left(\frac{2\pi v}{\lambda}\right)^2 \cos^2\left\{\frac{2\pi}{\lambda}(vt-x)\right\}.$$

The energy densities at '$t=0$',

$$\boxed{\begin{aligned} \frac{du}{dx} &= \frac{1}{2} T A^2 \left(\frac{2\pi}{\lambda}\right)^2 \cos^2\left(\frac{2\pi x}{\lambda}\right) \\ \& \quad \frac{dK}{dx} &= \frac{1}{2}\mu A^2 \left(\frac{2\pi v}{\lambda}\right)^2 \cos^2\left(\frac{2\pi x}{\lambda}\right) \end{aligned}}.$$

$$\therefore U = \int_0^\lambda \frac{du}{dx}\,dx = \frac{1}{2} T A^2 \left(\frac{2\pi}{\lambda}\right)^2 \int_0^\lambda \cos^2\left(\frac{2\pi x}{\lambda}\right) dx.$$

$$= \frac{\pi^2 A T^2}{\lambda}.$$

$$\therefore \boxed{U = \pi^2 A^2 \mu \nu^2 \lambda} \qquad \because \left[\begin{aligned} v &= \nu\lambda \\ v &= \sqrt{\frac{T}{\mu}} \end{aligned}\right].$$

Similarly,

$$K = \int_0^\lambda \frac{dK}{dx}\,dx = \frac{1}{2}\mu A^2 \left(\frac{2\pi v}{\lambda}\right)^2 \int_0^\lambda \cos^2\left(\frac{2\pi x}{\lambda}\right) dx.$$

$$\boxed{K = \pi^2 A^2 \mu \nu^2 \lambda} \longrightarrow \boxed{\text{same as } U \text{ for one complete } \lambda}$$

The total energy per wavelength is.

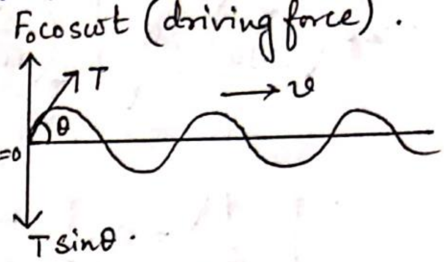
$$E_{total} = U + K = 2\pi^2 A^2 \mu \nu^2 \lambda.$$

$$\therefore \boxed{E_{total} \text{ per unit length of the string} = 2\pi^2 A^2 \mu \nu^2}.$$

Now to calculate the rate of transfer of energy in a string, suppose a harmonic driving force F is applied at the end $x=0$ of the string.

$$\boxed{F = -T\sin\theta}$$ [$\because$ force must be equal and opposite to the transverse component of tension]

or, $F = -T\tan\theta$ [$\because \theta$ is small]



$$\therefore\ F = -T\left(\frac{\partial y}{\partial x}\right)\bigg|_{x=0}.$$

$$\therefore\ \boxed{F = AT\left(\frac{2\pi}{\lambda}\right)\cos\left(\frac{2\pi vt}{\lambda}\right)}$$

The rate at which the energy is supplied to the string at $x=0$ or the input power at time 't' is,

$$\boxed{P(t) = \text{force} \times \text{velocity}}.$$

$$P(t) = F\left(\frac{\partial y}{\partial t}\right)\bigg|_{x=0}.$$

$$P(t) = A^2 T v\left(\frac{2\pi}{\lambda}\right)^2 \cos^2\left(\frac{2\pi vt}{\lambda}\right).$$

$$\boxed{P(t) = A^2 T v\left(\frac{2\pi}{\lambda}\right)^2 \cos^2\left(\frac{2\pi t}{\tau}\right)}$$ [$\because \frac{1}{\tau} = \frac{v}{\lambda}$]

The mean power input,

$$\langle P(t) \rangle = A^2 T v \left(\frac{2\pi}{\lambda}\right)^2 \frac{\int_0^{\tau} \cos^2\left(\frac{2\pi t}{\tau}\right) dt}{\int_0^{\tau} dt}$$

$$= \frac{1}{2} A^2 T v \left(\frac{2\pi}{\tau}\right)^2 .$$

$$\boxed{\langle P(t) \rangle = 2\pi^2 A^2 \mu v \nu^2} .$$

Thus, The time-averaged input power is equal to The total energy per unit length times The wave velocity (v). The energy flows along the string which acts as a medium for transport of energy. The speed of transport being equal to the wave velocity.