VIVEKANANDA COLLEGE
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NAAC ACCREDITED ‘A’ GRADE

Topic: Electrostatics and Magnetostatics

Course Title: Classical Electrodynamics

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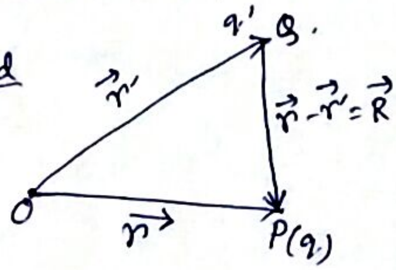
Name of the Department: Physics

Electrodynamics.

Electrostatics:- Coulomb's law:-

Force on charge $q(\vec{r})$ due to another point charge $q'(\vec{r}')$

$$= \vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qq'}{|\vec{r}-\vec{r}'|^3} \cdot (\vec{r}-\vec{r}') = \frac{1}{4\pi\epsilon_0} \frac{qq'}{R^3} \vec{R}, \text{ where, } |\vec{R}| = |\vec{r}-\vec{r}'|$$



Here $Q(\vec{r}')$ is the source point & $P(\vec{r})$ is the field point (the point of measurement).

Electric field at a point $P(\vec{r})$ due to a point charge $q'(\vec{r}')$ at Q $= \vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q'\hat{R}}{R^2}$

Electric field at a point $P(\vec{r})$ due to volume distribution of charge $\rho(\vec{r}') = \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{v'} \frac{\rho(\vec{r}')}{R^2} \hat{R}\, dv'$

We know that, $\vec{\nabla}\times\vec{E} = 0$ in electrostatics $\Rightarrow \vec{E} = -\vec{\nabla}\phi$

The work done against the field in moving a unit positive charge from a point 'a' to point 'b' $= W_{ab}$ = + work done against the field = - work done by the field $= -\int_a^b \vec{E}\cdot d\vec{r}$

$$= \int_a^b \vec{\nabla}\phi \cdot d\vec{r} = \int_a^b d\phi = \phi(b) - \phi(a)$$

If point 'a' is at infinity, then, $\phi(a) = \phi(\infty) = 0$

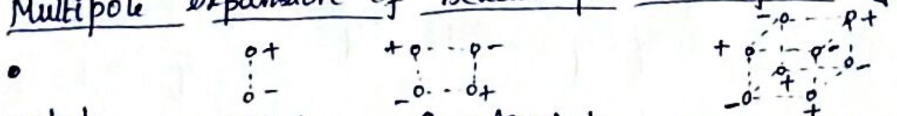
$$\therefore \phi(b) = -\int_\infty^b \vec{E}\cdot d\vec{r}.$$

Electrostatic potential at a point $P(\vec{r})$ due to a point charge q' at $Q(\vec{r}')$ =

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q'}{|\vec{r}-\vec{r}'|}$$

In the case of continuous charge distribution, $\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{v'} \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} dv'$

Multipole expansion of scalar potential of charge distribution:- [Reference: B. Ghosh]



Monopole Dipole Quadrupole Octapole.

If you are very far away from a localized charge distribution, it looks like a point charge and potential becomes approximately $\frac{1}{4\pi\epsilon_0}\frac{Q}{r}$. If Q is zero, then potential will be $\simeq 0$. Potential at a very large distance (r is very large) is pretty small if Q is not zero and this is the concept of multipole expansion.

1 Debye = 1D = 3.336×10^{-30} C.m.

So, in the case of multipole expansion, $|\vec{r}| >> |\vec{r}'| \Rightarrow$ very large distance between the source point and the point of measurement and for that $\rho \simeq 0$.

2.

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{v'} \frac{\rho(\vec{r}')dv'}{|\vec{r}-\vec{r}'|}$$

$$\frac{1}{R} = \frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{(r^2+r'^2 - 2.\vec{r}.\vec{r}')^{1/2}} = \frac{1}{r}\left[1+\frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta\right]^{-1/2}$$

$$= \frac{1}{r}(1+\delta)^{-1/2} \qquad \left[\because \delta = \frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta\right]$$

$$= \frac{1}{r}\left(1-\frac{1}{2}\delta + \frac{3}{8}\delta^2 - \frac{5}{16}\delta^3 + \ldots\right) = \frac{1}{r}\left(1-\frac{1}{2}\frac{r'^2}{r^2} + \frac{r'}{r}\cos\theta + \frac{3}{2}\frac{r'^2}{r^2}\cos^2\theta + \ldots\right)$$

$$= \frac{1}{r}\left[1+\frac{r'}{r}\cos\theta + \left(\frac{r'}{r}\right)^2\frac{1}{2}(3\cos^2\theta - 1) + \cdots\right]$$

$$\therefore \phi(\vec{r}) = \frac{1}{4\pi\epsilon_0 r}\int_{v'} \rho dv' + \frac{1}{4\pi\epsilon_0 r^2}\int_{v'} r'\cos\theta\,\rho dv' + \frac{1}{4\pi\epsilon_0 r^3}\int_{v'} r'^2\frac{1}{2}(3\cos^2\theta - 1)\rho dv' + \cdots$$

$$= \phi_0(\vec{r}) + \phi_1(\vec{r}) + \phi_2(\vec{r}) + \cdots\cdots$$

$$\phi_0(\vec{r}) = \frac{1}{4\pi\epsilon_0 r}\int_{v'} \rho dv' = \text{Monopole term}$$

$$\phi_1(\vec{r}) = \frac{1}{4\pi\epsilon_0 r^2}\int_{v'} r'\cos\theta\,\rho dv' \qquad \left[r'\cos\theta = \hat{r}.\vec{r}'; \quad p = \int_{v'} \vec{r}'\rho dv' \right.$$
= dipole moment of charge distribution]

$$= \frac{\hat{r}.\vec{p}}{4\pi\epsilon_0 r^2} = \text{dipole term.}$$

If total charge is zero, then $\vec{p}$ is independent of the choice of origin.

$$\phi_2(\vec{r}) = \frac{1}{4\pi\epsilon_0 r^3}\int_{v'} r'^2\frac{1}{2}(3\cos^2\theta - 1)\rho dv' = \text{Quadrupole term}$$

$$= \frac{1}{4\pi\epsilon_0 r^3}\cdot\frac{1}{2r^2}\int_{v'} r^2 r'^2(3\cos^2\theta - 1)\rho dv' = \frac{1}{4\pi\epsilon_0 . 2r^5}\int\left[3(\vec{r}.\vec{r}')^2 - r^2 r'^2\right]\rho dv'$$

Let, $\vec{r} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$.

$$\therefore (\vec{r}.\vec{r}')^2 = \left(\sum_{i=1}^{3} x_i x_i'\right)^2 = \sum_i\sum_j x_i x_j x_i' x_j'$$

$$r^2 r'^2 = r'^2\sum_i x_i^2 = r'^2\sum_i\sum_j x_i x_j \delta_{ij}, \text{ where, } \delta_{ij} = \begin{matrix} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j\end{matrix}$$

$$\therefore \phi_2(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i\sum_j \frac{x_i x_j}{2r^5}\int_{v'}(3x_i' x_j' - \delta_{ij} r'^2)\rho dv'$$

$$= \frac{1}{4\pi\epsilon_0}\sum_i\sum_j \frac{x_i x_j}{2r^5} Q_{ij}$$

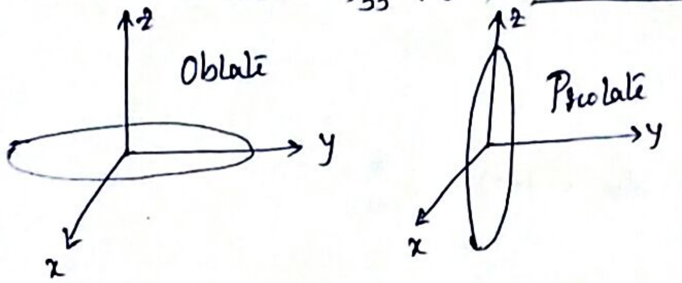
$$Q_{ij} = \int_{v'}\rho(3x_i' x_j' - \delta_{ij} r'^2)dv' = \text{Quadrupole moment tensor. } (i,j = 1,2,3)$$
$= Q_{ji}$

Out of these nine components, six are equal in pairs and hence, six components are distinct. Also, there is another property that the sum of the diagonal elements is zero, i.e; $Q_{11} + Q_{22} + Q_{33} = 0$. By proper choice of the co-ordinate system it is possible to make the 'off-diagonal' terms zero (i.e; $Q_{ij} = 0$ for $i \neq j$). If the charge distribution is symmetric about an axis, say z - axis, then all the quadrupole elements are expressible in terms of a single element $Q_{33} (= Q_{zz})$. For such a distribution, $Q_{11} = Q_{22} = -\frac{1}{2} Q_{33}$. Q_{33} is then called the 'quadrupole moment of the distribution'.

$$\therefore\ Q_{33} = \int_V (3z'^2 - r'^2)\rho\, dv.$$

For a charge distribution having symmetry about z - axis and elongated along this z - axis. $Q_{33} > 0 \Rightarrow$ prolate distribution.

Distribution which spreads more in the xy-plane and has symmetry about z - axis. $Q_{33} < 0 \Rightarrow$ oblate distribution.



If a charge q is at the origin, monopole term exists but p_i, Q_{ij} & higher order terms do not exist. Monopole moment does not depend on the position of the charge, so monopole moment remains same at everywhere as origin. But dipole moment at the origin is zero.

Multipole expansion in Spherical co-ordinate system :- In spherical polar co-ordinates (r, θ, ψ) Laplace's equation $\nabla^2\phi = 0$ have the form, [Ref: A.K. Raychaudhuri]

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \psi^2} = 0$$

We got, $\frac{1}{R} = \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\theta + \left(\frac{r'}{r}\right)^2 \cdot \frac{1}{2}(3\cos^2\theta - 1) + \cdots\right]$

In terms of Legendre polynomial, $\frac{1}{R} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta)$

$$\therefore\ \phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int_{V'}\rho(\vec{r}')\, r'^l\, P_l(\cos\theta)\, dV'$$

By addition theorem, $P_l(\cos\theta) = \frac{4\pi}{2l+1}\sum_{m=-l}^{+l} Y^*_{lm}(\theta', \psi')\, Y_{lm}(\theta, \psi)$

$Y_{lm}(\theta, \psi) = P_l^m(\cos\theta)\, e^{im\psi}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \Rightarrow$ These are orthonormal functions known as Spherical harmonics.

$$\therefore\ \phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{4\pi}{(2l+1)\, r^{l+1}}\left[\int_{V'}\rho(\vec{r}')\, r'^l\, Y^*_{lm}(\theta', \psi')\, dv'\right] Y_{lm}(\theta, \psi)$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{l, m}\frac{4\pi}{2l+1}\cdot\frac{q_{lm}\, Y_{lm}(\theta, \psi)}{r^{l+1}}$$

q_{lm} = Multipole moments $= \int_{V'}\rho(\vec{r}')\, r'^l\, Y^*_{lm}(\theta', \psi')\, dv'$

4.

$$q_{l,-m} = \int_{V'} \rho(\vec{r}')\, r'^{l}\, Y^*_{l,-m}(\theta',\psi')\, dv'$$

For each l value, m has $(2l+1)$ values from $-l$ to $+l$.

$l=0$ = monopole moments.

$l=1$ = dipole "

$l=2$ = quadrupole "

[Note: $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \Rightarrow -\nabla^2\phi = \rho/\epsilon_0 \Rightarrow \nabla^2\phi = -\rho/\epsilon_0 \Rightarrow$ Poisson's equⁿ.
Where there is no charge, $\rho=0$. Therefore, $\nabla^2\phi = 0 \Rightarrow$ Laplace's equⁿ.]

[Note: $P_l^m(x) = (-1)^m (1-x^2)^{m/2} \dfrac{d^m}{dx^m} P_l(x)$

$P_l^{-m}(x) = (-1)^m \dfrac{(l-m)!}{(l+m)!} P_l^m(x)$]

$\boxed{Y_{l,-m}(\theta,\psi) = (-1)^m\, Y^*_{lm}(\theta,\psi)}$ Remember

$$\therefore q_{l,-m} = \int_{V'} (-1)^m \rho(\vec{r}')\, r'^{l}\, Y_{lm}(\theta',\psi')\, dv' = (-1)^m q^*_{lm}$$

$$\therefore \boxed{q_{l,-m} = (-1)^m q^*_{lm}}$$

Monopole term:- $l=0, m=0 \Rightarrow Y_{00}(\theta,\psi) = P_0^0(\cos\theta)\cdot e^{i\cdot 0\cdot\psi}\sqrt{\dfrac{1}{4\pi}\dfrac{0!}{0!}} = 1.1.\dfrac{1}{\sqrt{4\pi}}$
$= \dfrac{1}{\sqrt{4\pi}}$; $Y^*_{00}(\theta',\psi') = \dfrac{1}{\sqrt{4\pi}}$.

$q_{lm} = q_{00} = \int \rho(\vec{r}')\cdot r'^0 \cdot \dfrac{1}{\sqrt{4\pi}}\, dv' = \dfrac{1}{\sqrt{4\pi}}\int \rho(\vec{r}')\, dv = \dfrac{1}{\sqrt{4\pi}}\cdot Q$; Q: total charge.

$\phi_0(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\cdot\dfrac{4\pi}{2.0+1}\cdot\dfrac{1}{r^{0+1}}\cdot q_{00}Y_{00} = \dfrac{1}{4\pi\epsilon_0}\cdot\dfrac{4\pi}{r}\cdot\dfrac{1}{\sqrt{4\pi}}\cdot\dfrac{Q}{\sqrt{4\pi}} = \dfrac{Q}{4\pi\epsilon_0 r}$

Potential due to a point charge Q

Dipole term:- $l=1 \Rightarrow m = 1, 0, -1$.

i) $l=1, m=0 \Rightarrow Y_{10}(\theta,\psi) = P_1^0(\cos\theta)\cdot e^{i\cdot 0\cdot\psi}\sqrt{\dfrac{2.1+1}{4\pi}\dfrac{1!}{1!}} = \cos\theta\cdot 1\cdot\sqrt{\dfrac{3}{4\pi}} = \sqrt{\dfrac{3}{4\pi}}\cos\theta$

$Y^*_{10}(\theta',\psi') = \sqrt{\dfrac{3}{4\pi}}\cos\theta'$; $q_{10} = \int \rho(\vec{r}')(r')\sqrt{\dfrac{3}{4\pi}}\cos\theta'\, dv' = \sqrt{\dfrac{3}{4\pi}}\int r'\cos\theta'\rho(\vec{r}')\, dv'$
$= \sqrt{\dfrac{3}{4\pi}}\, p_z$.

$\therefore \dfrac{4\pi}{2l+1}\dfrac{q_{lm}Y_{lm}}{r^{l+1}} = \dfrac{4\pi}{3}\cdot\dfrac{1}{r^2}\sqrt{\dfrac{3}{4\pi}}\, p_z\cdot\sqrt{\dfrac{3}{4\pi}}\cos\theta = \dfrac{1}{r^2}\, p_z\cdot\dfrac{r\cos\theta}{r}$
$= \dfrac{1}{r^3}\, z p_z$.

[Note: In spherical co-ordinate system, $x = x_1 = r\sin\theta\cos\psi$, $y = x_2 = r\sin\theta\sin\psi$,
$z = x_3 = r\cos\theta$].

ii) $l=1, m=1 \Rightarrow Y_{11}(\theta,\psi) = P_1^1(\cos\theta)\, e^{i\cdot 1\cdot\psi}\sqrt{\dfrac{3}{4\pi}\cdot\dfrac{0!}{2!}} = (-\sin\theta)\cdot e^{i\psi}\sqrt{\dfrac{3}{8\pi}}$.
$= (-\sin\theta)(\cos\psi + i\sin\psi)\sqrt{\dfrac{3}{8\pi}} = -\sqrt{\dfrac{3}{8\pi}}\,\dfrac{1}{r}(r\sin\theta\cos\psi + ir\sin\theta\sin\psi)$
$= -\sqrt{\dfrac{3}{8\pi}}\,\dfrac{1}{r}(x+iy)$

5.

$$Y_{11}^{*}(\theta',\varphi') = -\sqrt{\frac{3}{8\pi}}\,(\sin\theta'\cos\varphi' - i\sin\theta'\sin\varphi')$$

$$q_{11} = -\int \rho(\vec{r}')(r')^{1}\sqrt{\frac{3}{8\pi}}\,(\sin\theta'\cos\varphi' - i\sin\theta'\sin\varphi')\,dv'$$

$$= -\sqrt{\frac{3}{8\pi}}\int (r'\sin\theta'\cos\varphi' - ir'\sin\theta'\sin\varphi')\,\rho(\vec{r}')\,dv'$$

$$= -\sqrt{\frac{3}{8\pi}}\,(p_x - ip_y)$$

$$\therefore \frac{4\pi}{2l+1}\,\frac{q_{lm}Y_{lm}}{r^{l+1}} = \frac{4\pi}{3}\,\frac{1}{r^2}\left[-\sqrt{\frac{3}{8\pi}}\,(p_x - ip_y)\right]\left[-\sqrt{\frac{3}{8\pi}}\,\frac{1}{r}(x+iy)\right]$$

$$= \frac{1}{2r^3}(x+iy)(p_x - ip_y).$$

iii) $l=1,\ m=-1,$ $\quad Y_{1,-1}(\theta,\varphi) = (-1)^{1}\,Y_{11}^{*}(\theta,\varphi) = (-1)\left[-\sqrt{\frac{3}{8\pi}}\,\frac{1}{r}(x-iy)\right]$

$$= \sqrt{\frac{3}{8\pi}}\,\frac{1}{r}(x-iy)$$

$$q_{1,-1} = (-1)^{1}q_{11}^{*} = (-1)\left[-\sqrt{\frac{3}{8\pi}}\,(p_x+ip_y)\right] = \sqrt{\frac{3}{8\pi}}\,(p_x+ip_y)$$

$$\therefore \frac{4\pi}{2l+1}\,\frac{q_{lm}Y_{lm}}{r^{l+1}} = \frac{4\pi}{3}\cdot\frac{1}{r^2}\left[\sqrt{\frac{3}{8\pi}}\,(p_x+ip_y)\right]\left[\sqrt{\frac{3}{8\pi}}\,\frac{1}{r}(x-iy)\right]$$

$$= \frac{1}{2r^3}(x-iy)(p_x+ip_y)$$

$$\phi_1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\cdot\left[\frac{1}{r^3}zp_z + \frac{1}{2r^3}(x+iy)(p_x-ip_y) + \frac{1}{2r^3}(x-iy)(p_x+ip_y)\right]$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{1}{2r^3}\left[2zp_z + (xp_x + iyp_x - ixp_y + yp_y + xp_x - iyp_x + ixp_y + yp_y)\right]$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{1}{2r^3}\left[2(zp_z + xp_x + yp_y)\right] = \frac{1}{4\pi\epsilon_0 r^3}\,\vec{p}\cdot\vec{r} = \text{dipole potential.}$$

$\vec{p}$ = dipole moment = $p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$

Similarly, quadrupole potential term can be evaluated for $l=2$.

Energy of multipoles in an external field (Interaction of a charge distribution with potential):- Let us consider an external field with potential $\phi(\vec{r})$, then the interaction energy of the charge distribution $\rho(\vec{r})$ with the potential, $W = \int \rho(\vec{r})\,\phi(\vec{r})\,dv$

Expanding $\phi(\vec{r})$ in a Taylor series, ~~$\phi(\vec{r})$~~ with respect to $\vec{r}=0$,

$$\phi(\vec{r}) = \phi(0) + \sum_i \left(\frac{\partial\phi}{\partial x_i}\right)_0 x_i + \frac{1}{2}\sum_i\sum_j \left(\frac{\partial^2\phi}{\partial x_i\partial x_j}\right)_0 x_i x_j + \cdots$$

$$= \phi_0 - \sum_i E_{i0}x_i - \frac{1}{2}\sum_{i,j}\left(\frac{\partial E_j}{\partial x_i}\right)_0 x_i x_j + \cdots \quad \left[\because \frac{\partial\phi}{\partial x_i} = -E_i,\ \frac{\partial^2\phi}{\partial x_i\partial x_j} = -\frac{\partial}{\partial x_i}(E_j)\right]$$

$$\therefore \text{Energy } W = \int \rho(\vec{r})\,\phi(\vec{r})\,dv = \int \phi_0\,\rho(\vec{r})\,dv + \sum_i -\left[\int \rho(\vec{r})\,x_i\,dv\right]E_{i0}$$

$$+ \sum_{ij}\left(-\frac{1}{2}\right)\left[\int \rho(\vec{r})\,x_i x_j\,dv\right]\left(\frac{\partial E_j}{\partial x_i}\right)_0 + \cdots$$

6.

Now,

For very large value of $\vec{r}$, $\rho = 0$

$$\therefore \vec{\nabla} \cdot \vec{E} = 0 \Rightarrow \sum_i \frac{\partial E_i}{\partial x_i} = 0 \Rightarrow \sum_i \sum_j \left(\frac{\partial E_j}{\partial x_i}\right)_0 \delta_{ij} = 0 \Rightarrow \frac{r^2}{6} \sum_i \sum_j \left(\frac{\partial E_j}{\partial x_i}\right)_0 \delta_{ij} = 0$$

$$\therefore W = q\phi_0 - \sum_i p_i E_{i0} - \frac{1}{6} \sum_{i,j} \left[\int (3x_i x_j - r^2 \delta_{ij}) \rho(\vec{r}) dv\right] \left(\frac{\partial E_j}{\partial x_i}\right)_0 + \cdots$$

$$= q\phi_0 - \vec{p} \cdot \vec{E}(0) - \frac{1}{6} \sum_{i,j} \left(\frac{\partial E_j}{\partial x_i}\right)_0 Q_{ij} = W_0 + W_1 + W_2 + \cdots$$

$$Q_{ij} = \int \rho(\vec{r}) (3x_i x_j - r^2 \delta_{ij}) \, dv.$$

W_0 = monopole energy ; W_1 = dipole energy ; W_2 = quadrupole energy.

From previous study, we know that, $Q_{xx} + Q_{yy} + Q_{zz} = 0$

For axial symmetry along z-axis, $Q_{xx} = Q_{yy} = -\frac{Q_{zz}}{2}$.

$$\sum_i \frac{\partial E_i}{\partial x_i} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \Rightarrow \frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = -\frac{1}{2}\frac{\partial E_z}{\partial z}.$$

Let us calculate quadrupole energy W_2.

$$W_2 = -\frac{1}{6}\left[Q_{xx}\frac{\partial E_x}{\partial x} + Q_{yy}\frac{\partial E_y}{\partial y} + Q_{zz}\frac{\partial E_z}{\partial z}\right] = -\frac{1}{6}\left[\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + 1\right] Q_{zz}\frac{\partial E_z}{\partial z}$$

$$= -\frac{1}{6}\left[\frac{1}{4} + \frac{1}{4} + 1\right] Q_{zz}\frac{\partial E_z}{\partial z} = -\frac{1}{6} \times \frac{6}{4} Q_{zz}\frac{\partial E_z}{\partial z} = -\frac{1}{4} Q_{zz}\frac{\partial E_z}{\partial z}.$$

For a point charge, $\phi = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$.

$$E_z = -\frac{\partial \phi}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right) = -\frac{1}{4\pi\epsilon_0}\cdot\left(\frac{-q}{r^2}\right)\cdot\frac{\partial r}{\partial z} = \frac{q}{4\pi\epsilon_0 r^2}\cdot\frac{z}{r} = \frac{qz}{4\pi\epsilon_0 r^3}$$

$$\frac{\partial E_z}{\partial z} = \frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r^3} - \frac{3z}{r^4}\cdot\frac{z}{r}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r^3} - \frac{3z^2}{r^5}\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r^3} - \frac{3r^2\cos^2\theta}{r^5}\right] \quad [\because z = r\cos\theta]$$

$$= \frac{q}{4\pi\epsilon_0 r^3}(1 - 3\cos^2\theta)$$

$$\therefore W_2 = -\frac{1}{4}Q_{zz}\frac{\partial E_z}{\partial z} = -\frac{1}{4}Q_{zz}\cdot\frac{q}{4\pi\epsilon_0 r^3}(1 - 3\cos^2\theta) = \frac{q Q_{zz}}{16\pi\epsilon_0 r^3}(1 - 3\cos^2\theta)$$

Force $\vec{F} = \vec{\nabla} W$

Torque $\vec{\tau} = \vec{r} \times \vec{F}$.

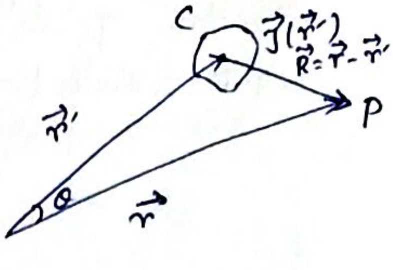
7.

~~Multipole expansion of~~ [Ref: B. Ghosh]

Magnetostatics :- Biot - Savart law :- $d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{l} \times \vec{R}}{R^3} = \frac{\mu_0}{4\pi} \frac{\vec{J} \times \vec{R}}{R^3} dV$

$\vec{\nabla} \cdot \vec{B} = 0$; $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}')}{R} dV'$ [$\because \vec{B} = \vec{\nabla} \times \vec{A}$; $\vec{A}$ = vector potential].

Multipole expansion of the vector potential due to a localized current-distribution :- If the distance between the current distribution and the point of potential measurement is sufficiently large, the series will be dominated by the lowest non-vanishing contribution and the higher terms can be ignored.



$$\frac{1}{R} = \frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\theta)$$

Accordingly, the vector potential of a current loop can be written

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \Big[\underbrace{\frac{1}{r} \oint d\vec{l}'}_{\text{monopole term}} + \underbrace{\frac{1}{r^2} \oint r' \cos\theta \, d\vec{l}'}_{\text{dipole term}} + \underbrace{\frac{1}{r^3} \oint (r')^2 \left(\frac{3}{2}\cos^2\theta - \frac{1}{2} \right) d\vec{l}'}_{\text{quadrupole term.}} + \cdots \Big]$$

Magnetic monopole term is always zero, for the integral is just the total vector displacement around a closed loop c : $\oint d\vec{l}' = 0 \Rightarrow$ No that implies no. magnetic monopole exist in nature.

~~Quadrupole term~~ → **Dipole term :-**

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\theta \, d\vec{l}' = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{r} \cdot \vec{r}') d\vec{l}'$$

$$\hat{r} \times (d\vec{l}' \times \vec{r}') = d\vec{l}'(\hat{r} \cdot \vec{r}') - \vec{r}'(\hat{r} \cdot d\vec{l}') \quad \cdots\cdots (1)$$

For a small change in $\vec{r}'$, $d[\vec{r}'(\hat{r} \cdot \vec{r}')] = d\vec{r}'(\hat{r} \cdot \vec{r}') + \vec{r}'(\hat{r} \cdot d\vec{r}')$ $\cdots$ (2)

Now, the position vector of the current loop is $\vec{r}'$ from origin. $d\vec{l}'$ is the small ~~length~~ elemental length of the current loop. So, we can say, $d\vec{r}' = d\vec{l}'$

(2) gives, $d[\vec{r}'(\hat{r} \cdot \vec{r}')] = d\vec{l}'(\hat{r} \cdot \vec{r}') + \vec{r}'(\hat{r} \cdot d\vec{l}')$ $\cdots$ (3).

(1) + (3) $\Rightarrow$ $\hat{r} \times (d\vec{l}' \times \vec{r}') + d[\vec{r}'(\hat{r} \cdot \vec{r}')] = 2\, d\vec{l}'(\hat{r} \cdot \vec{r}') = 2(\hat{r} \cdot \vec{r}') d\vec{l}'$ $\cdots$ (4)

$\oint d[\vec{r}'(\hat{r} \cdot \vec{r}')] = 0$ [$\because$ line integral about a closed loop is zero always].

(4) $\Rightarrow$ $\oint (\hat{r} \cdot \vec{r}') d\vec{l}' = \frac{1}{2} \oint \hat{r} \times (d\vec{l}' \times \vec{r}')$

$$\therefore \vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{r} \cdot \vec{r}') d\vec{l}' = \frac{\mu_0 I}{4\pi r^2} \cdot \frac{1}{2} \oint \hat{r} \times (d\vec{l}' \times \vec{r}')$$

$$= \frac{\mu_0 I}{4\pi r^2} \cdot \frac{1}{2} \oint -(d\vec{l}' \times \vec{r}') \times \hat{r} = \frac{\mu_0 I}{4\pi r^2} \frac{1}{2} \oint (\vec{r}' \times d\vec{l}') \times \hat{r}$$

Now, $\frac{1}{2}\oint_{c} (\vec{r}' \times d\vec{l}') = \vec{S} =$ area of the loop. $\vec{S}$ points $\perp^r$ to the plane of the loop.

$\vec{m} = I.\vec{S} =$ magnetic dipole moment of the loop.

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi r^2}(\vec{m}\times\hat{r}) = \frac{\mu_0}{4\pi r^3}(\vec{m}\times\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}.$$

Direction of $\vec{m}$ is given by the direction of movement of a right-handed screw when rotated in the direction of current.

Maxwell's Equations

i) $\vec{\nabla}.\vec{E} = \rho/\epsilon_0$ .... ①

ii) $\vec{\nabla}.\vec{B} = 0$ ..... ②

iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ .... ③

iv) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$ .... ④

$\vec{B}$ is divergence always $\Rightarrow \vec{\nabla}.\vec{B} = 0$. We can write, $\vec{B} = \vec{\nabla}\times\vec{A}$ ... ⑤

In electrostatics, $\vec{\nabla}\times\vec{E} = 0$ allowed us to write $\vec{E}$ as the gradient of a scalar potential: $\vec{E} = -\vec{\nabla}\phi$.

In electrodynamics, this is no longer possible, because the curl of $\vec{E}$ is non-zero.

③ $\rightarrow \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$

$\Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0 \Rightarrow$ It can be written as gradient of a scalar.

$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi \Rightarrow \vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$ ..... ⑥

① $\rightarrow \vec{\nabla}.\vec{E} = \rho/\epsilon_0 \Rightarrow \vec{\nabla}.\left(-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right) = \rho/\epsilon_0 \Rightarrow \nabla^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) = -\rho/\epsilon_0$ ... ⑦

④ $\rightarrow \vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

$\Rightarrow \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right)$

$\Rightarrow \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A} + \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} + \mu_0\epsilon_0\vec{\nabla}\frac{\partial\phi}{\partial t} = \mu_0\vec{J}$

$\Rightarrow \left(\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}.\vec{A} + \mu_0\epsilon_0\frac{\partial\phi}{\partial t}\right) = -\mu_0\vec{J}$ ...... ⑧

Gauge transformations :- Suppose we have two sets of potentials, $(\phi,\vec{A})$ & $(\phi',\vec{A}')$ which correspond to the same electric & magnetic fields. $(\vec{E}' = \vec{E}$ & $\vec{B}' = \vec{B})$ [Ref: Griffiths]

Let, $\vec{A}' = \vec{A} + \vec{\alpha}$ and $\phi' = \phi + \vec{\beta}$

$\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times(\vec{A} + \vec{\alpha}) = \vec{\nabla}\times\vec{A} + \vec{\nabla} + \vec{\alpha} = \vec{B} + (\vec{\nabla}\times\vec{\alpha})$.

Now, $\vec{B}' = \vec{B} \Rightarrow \vec{\nabla}\times\vec{\alpha} = 0 \Rightarrow \vec{\alpha}$ must be the gradient of a scalar (λ).

$\therefore \boxed{\vec{\alpha} = \vec{\nabla}\lambda}$ .... ⑨

$$\vec{E}' = -\vec{\nabla}\phi' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}(\phi+\beta) - \frac{\partial}{\partial t}(\vec{A}+\vec{\nabla}\lambda) = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\beta - \vec{\nabla}\frac{\partial \lambda}{\partial t}$$
$$= \vec{E} - \vec{\nabla}\beta - \vec{\nabla}\frac{\partial \lambda}{\partial t}$$

Now, $\vec{E}' = \vec{E}$.

$$\therefore \quad \vec{\nabla}\beta + \vec{\nabla}\frac{\partial\lambda}{\partial t} = 0 \Rightarrow \boxed{\beta = -\frac{\partial \lambda}{\partial t}} \quad \cdots (10)$$

$\therefore$ we get,
$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda \quad \cdots (11)$$
$$\phi' = \phi - \frac{\partial\lambda}{\partial t} \quad \cdots (12)$$

(11) & (12) these transformations do not affect the physical quantities $\vec{E}$ & $\vec{B}$. Such changes in ϕ & $\vec{A}$ are called <u>gauge transformations</u>.

[Note: Equations (2) & (3) are <u>homogeneous</u> but (1) & (4) are <u>inhomogeneous</u> equations. So what we are trying to do that, using (2) & (3) we get $\vec{E}$ & $\vec{B}$ in the forms of ϕ & $\vec{A}$, and using those values we have to refresh equations (1) & (4) which are now converted into (7) & (8) and we will do it now].

We already know about coulomb gauge, $\vec{\nabla}\cdot\vec{A} = 0$ in magnetostatics. But in electrodynamics, situation is different. We will know about two most popular gauges.

<u>Lorentz gauge</u> :- (7) $\longrightarrow \nabla^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\epsilon_0$ [Ref: Griffiths & Jackson-3rd ed.]

$$\Rightarrow \nabla^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\epsilon_0$$
$$\Rightarrow \left(\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\right) + \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = -\rho/\epsilon_0 \quad \cdots (13)$$

Now, $\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \equiv \Box^2 \Rightarrow$ d'Alembertian operator

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = L \Rightarrow \text{Lorentz condition} \quad \cdots (14)$$

$$(13) \Rightarrow \quad \Box^2\phi + \frac{\partial L}{\partial t} = -\rho/\epsilon_0 \quad \cdots (15)$$

$$(8) \Rightarrow \left(\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = -\mu_0\vec{J}$$
$$\Rightarrow \Box^2\vec{A} - \vec{\nabla}L = -\mu_0\vec{J} \quad \cdots (16)$$

In Lorentz condition $L = 0 \Rightarrow \vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0 \Rightarrow \vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t} \quad \cdots (17)$

$$(15) \Rightarrow \Box^2\phi = -\rho/\epsilon_0 \quad \cdots (18)$$
$$(16) \Rightarrow \Box^2\vec{A} = -\mu_0\vec{J} \quad \cdots (19)$$

Let us see 'Lorentz gauge transformation' ($L' = L$).

10.

$$L' = \vec{\nabla}.\vec{A}' + \frac{1}{c^2}\frac{\partial \phi'}{\partial t} = 0 \Rightarrow \vec{\nabla}.(\vec{A}+\vec{\nabla}\lambda) + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\phi - \frac{\partial \lambda}{\partial t}\right) = 0$$

$$\Rightarrow \vec{\nabla}.\vec{A} + \nabla^2\lambda + \frac{1}{c^2}\frac{\partial \phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2\lambda}{\partial t^2} = 0 \Rightarrow \vec{\nabla}.\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla^2\lambda - \frac{1}{c^2}\frac{\partial^2\lambda}{\partial t^2} = 0$$

$$\Rightarrow L + \nabla^2\lambda - \frac{1}{c^2}\frac{\partial^2\lambda}{\partial t^2} = 0 \Rightarrow \boxed{\nabla^2\lambda - \frac{1}{c^2}\frac{\partial^2\lambda}{\partial t^2} = 0} \Rightarrow$$ Wave equation for λ. .... (20)

Coulomb gauge :- $\vec{\nabla}.\vec{A} = 0$ .... (21)

(7) $\nabla^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) = -\rho/\epsilon_0 \Rightarrow \nabla^2\phi = -\rho/\epsilon_0$ .... (22)

$$\Rightarrow \nabla^2\phi(\vec{r},t) = -\rho(\vec{r},t)/\epsilon_0 \Rightarrow \phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int_{V'}\frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}\,dv' \quad \ldots \boxed{22\text{-}A}.$$

It is the solution of the equation (22).
$\phi(\vec{r},t)$ is the Coulomb potential due to charge density $\rho(\vec{r}',t)$. This is the origin of the name 'Coulomb gauge'.

(8) $\Rightarrow \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}.\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = -\mu_0\vec{J}$

$$\Rightarrow \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \frac{1}{c^2}\vec{\nabla}\frac{\partial\phi}{\partial t} = -\mu_0\vec{J}$$

$$\Rightarrow \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \frac{1}{c^2}\vec{\nabla}\frac{\partial\phi}{\partial t} = -\mu_0\left(\vec{J} - \epsilon_0\vec{\nabla}\frac{\partial\phi}{\partial t}\right)$$

$$= -\mu_0(\vec{J} - \vec{J}_L) = -\mu_0\vec{J}_t \quad \ldots (23)$$

$\vec{J}_L = \epsilon_0\vec{\nabla}\frac{\partial\phi}{\partial t}$ = irrotational or longitudinal current density.... (24)

$\vec{J}_t = \vec{J} - \vec{J}_L$ = transverse or solenoidal current density.

$\vec{J}$ = total current density = $\vec{J}_L + \vec{J}_t \Rightarrow \vec{J}$ can be splitted to longitudinal and transverse components. .... (25)

(23) $\Rightarrow \Box^2\vec{A} = -\mu_0\vec{J}_t$ .... (26)

In can be seen that, $\vec{J}_L$ is gradient of a scalar, so its curl should be zero. $\therefore$ $\vec{\nabla}\times\vec{J}_L = 0$ but $\vec{\nabla}.\vec{J}_L \neq 0$ ... (27)

In the case of $\vec{J}_t \Rightarrow \vec{\nabla}.\vec{J}_t = 0$ and $\vec{\nabla}\times\vec{J}_t \neq 0$ ... (28)

[Let us know about 'Helmholtz's theorem' :- Consider a vector $\vec{F}(\vec{r})$.
Let, $\vec{\nabla}.\vec{F} = D$, where, D is a scalar $D(\vec{r})$ ... (29)
Let, $\vec{\nabla}\times\vec{F} = \vec{C}$, " $\vec{C}$ " " vector $\vec{C}(\vec{r})$ ... (30)
'Helmholtz's theorem' states that any sufficiently smooth rapidly decaying vector field in 3-D can be resolved into the sum of an irrotational (curl-free) vector field and a solenoidal (divergence-free) vector field.
If $\vec{F}(\vec{r})$ is a rapidly decaying vector, then we can write,

$$\vec{F} = -\vec{\nabla}\psi + \vec{\nabla}\times\vec{G} \quad \ldots (31)$$

where, $-\vec{\nabla}\psi$ is the curl-free component and $\vec{\nabla}\times\vec{G}$ is the ~~irrotational~~ divergence-free component.]

We can write, $\psi(\vec{r}) \equiv \frac{1}{4\pi}\int \frac{D(\vec{r}')}{|\vec{r}-\vec{r}'|}dv' = \frac{1}{4\pi}\int \frac{\vec{\nabla}'\cdot\vec{F}}{|\vec{r}-\vec{r}'|}dv'$ ...... (32)

& $\vec{G}(\vec{r}) \equiv \frac{1}{4\pi}\int \frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|}dv' = \frac{1}{4\pi}\int \frac{\vec{\nabla}'\times\vec{F}}{|\vec{r}-\vec{r}'|}dv'$ ..... (33)]

Now, we come back to current density $\vec{J}$.

$\boxed{22\text{-}A} \Rightarrow \phi = \frac{1}{4\pi\epsilon_0}\int_{v'} \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}dv' \Rightarrow \frac{\partial\phi}{\partial t} = \frac{1}{4\pi\epsilon_0}\int \frac{\partial\rho/\partial t}{|\vec{r}-\vec{r}'|}dv'$ .... (34)

Continuity equation :- $\vec{\nabla}'\cdot\vec{J} + \partial\rho/\partial t = 0$

$\Rightarrow \partial\rho/\partial t = -\vec{\nabla}'\cdot\vec{J}$

[$\vec{\nabla}'$ is used in place of $\vec{\nabla}$ because $\vec{J}$ is the source field with position vector $\vec{r}'(x',y',z')$, so, $\vec{\nabla}'$ is also for (x',y',z')]

$\therefore$ (34) $\Rightarrow \frac{\partial\phi}{\partial t} = -\frac{1}{4\pi\epsilon_0}\int \frac{\vec{\nabla}'\cdot\vec{J}}{|\vec{r}-\vec{r}'|}dv'$

(24) $\Rightarrow \vec{J}_L = \epsilon_0\vec{\nabla}\frac{\partial\phi}{\partial t} = -\frac{1}{4\pi}\vec{\nabla}\int \frac{\vec{\nabla}'\cdot\vec{J}}{|\vec{r}-\vec{r}'|}dv' = -\vec{\nabla}\psi$ .... (35)

Comparing, (32) & (35) we get $\vec{F} = \vec{J} = \vec{J}_L + \vec{J}_t = -\vec{\nabla}\psi + \vec{\nabla}\times\vec{G}$ ..... (36)

$\therefore$ (33) $\quad \vec{G}(\vec{r}) = \frac{1}{4\pi}\int \frac{\vec{\nabla}'\times\vec{J}}{|\vec{r}-\vec{r}'|}dv'$

(36) $\Rightarrow \vec{J}_t = \vec{\nabla}\times\vec{G} = \frac{1}{4\pi}\vec{\nabla}\times\int \frac{\vec{\nabla}'\times\vec{J}}{|\vec{r}-\vec{r}'|}dv'$ .... (37)

(26) $\Rightarrow \Box^2\vec{A} = \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}_t$

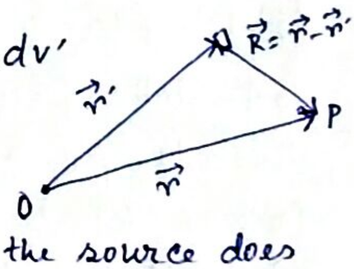
$\therefore$ The source for the wave equation for $\vec{A}$ can be expressed entirely in terms of the transverse current. So that, Coulomb gauge is also known as 'transverse gauge'. This gauge has another name, 'radiation gauge'. $\Rightarrow$ Find the reason

Retarded Potentials [Ref: Griffiths]

In the static case, $\Box^2\phi = -\rho/\epsilon_0 \Rightarrow \nabla^2\phi = -\rho/\epsilon_0 \left[\because \frac{\partial^2\phi}{\partial t^2} = 0\right]$

& $\Box^2\vec{A} = -\mu_0\vec{J} \Rightarrow \nabla^2\vec{A} = -\mu_0\vec{J}$

$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}dv'$; $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}dv'$



where, $R = |\vec{r}-\vec{r}'|$ = distance from the source point $\vec{r}'$ to the field point $\vec{r}$. e-m 'news' travels at the speed of light. In non-static case, present status of the source does not matter, but its condition at some earlier time t_r (retarded time), when the 'message' left matters. Since the message travels R distance in t_r time, $t_r \equiv t - R/c$ = Retarded time.

$\therefore$ For non-static sources, $\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)}{R}dv'$

$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)}{R}dv'$

Here, $\rho(\vec{r}', t_r) \Rightarrow$ charge density at the point $\vec{r}'$ at the retarded time t_r.
$\vec{J}(\vec{r}', t_r) \Rightarrow$ current " " " " " " " " " " ".

Because the integrands are evaluated at the retarded time, these are called retarded potentials $[\phi(\vec{r}, t), \vec{A}(\vec{r}, t)]$.

Incidentally, advanced potentials $\phi_a(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}', t_a)}{R} dv'$ & $\vec{A}_a(\vec{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}', t_a)}{R} dv'$ in which charge & current densities are evaluated at the advanced time $t_a \equiv t + R/c$.

Note : $\vec{\nabla} t_r = \vec{\nabla}(t - R/c) = -\frac{1}{c}\vec{\nabla} R$

Maxwell's Stress Tensor [Ref: Panofsky & Phillips, Griffiths & Jackson]

If we consider that the force acting on a given volume is transmitted across the elements of surface bounding that volume, then this transmitting force can be formulated in terms of a quantity known as the stress tensor T. The $\alpha\beta$-th component of $T_{\alpha\beta}$ of the stress tensor T is so constituted that the α-th component dF_α of the force $d\vec{F}$ transmitted across a surface element $d\vec{s}$ whose component in the β-th direction is dS_β, $dF_\alpha = \sum_{\beta=1}^{3} T_{\alpha\beta} dS_\beta$

$\therefore F_\alpha = \int dF_\alpha = \int \sum_\beta T_{\alpha\beta}\, dS_\beta$

Force F_α in terms of volume force (force per unit volume) $F_{v\alpha}$:-

$$F_\alpha = \int dF_\alpha = \int F_{v\alpha}\, dV = \int \sum_\beta T_{\alpha\beta} dS_\beta = \int \sum_\beta \frac{\partial T_{\alpha\beta}}{\partial x_\beta} dV \Rightarrow \boxed{F_{v\alpha} = \sum_\beta \frac{\partial T_{\alpha\beta}}{\partial x_\beta}}$$

$\Downarrow$

Volume force in terms of tensor divergence.

$T \Rightarrow$ surface stress tensor defines the stress transmitted by the field across the surface of the volume.

Electrostatic case :- According to Gauss's theorem in electrostatics,

$\int \vec{E}\cdot d\vec{s} = q/\epsilon_0 \Rightarrow \vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \Rightarrow \rho = \epsilon_0(\vec{\nabla}\cdot\vec{E})$ ..... ①

Electrostatic force on a point charge $= q\vec{E}$.

Similarly, for a charge distribution with density $\rho(\vec{r})$, force density (volume force) $= \rho\vec{E} = \vec{F_v} =$ force per unit volume ..... ②

② $\Rightarrow \vec{F_v} = \rho\vec{E} = \epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} \Rightarrow F_{v\alpha} = \epsilon_0\left(\sum_\beta \frac{\partial E_\beta}{\partial x_\beta}\right) E_\alpha$ ..... ③

Now, $\frac{\partial}{\partial x_\beta}(E_\alpha E_\beta) = \frac{\partial E_\alpha}{\partial x_\beta} E_\beta + \frac{\partial E_\beta}{\partial x_\beta} E_\alpha$.

③ $\Rightarrow F_{v\alpha} = \epsilon_0 \sum_\beta \left[\frac{\partial}{\partial x_\beta}(E_\alpha E_\beta) - \frac{\partial E_\alpha}{\partial x_\beta} E_\beta\right]$ ..... ④

In static case, $\vec{\nabla}\times\vec{E} = 0 \Rightarrow \frac{\partial E_\beta}{\partial x_\alpha} = \frac{\partial E_\alpha}{\partial x_\beta}$ ...... ⑤

Substituting ⑤ in ④ $\Rightarrow F_{v\alpha} = \epsilon_0 \sum_\beta \left[\frac{\partial}{\partial x_\beta}(E_\alpha E_\beta) - \frac{\partial E_\beta}{\partial x_\alpha} E_\beta\right]$

13.

$$F_{v\alpha} = \epsilon_0 \sum_\beta \left[\frac{\partial}{\partial x_\beta}(E_\alpha E_\beta) - \frac{1}{2}\frac{\partial}{\partial x_\alpha}(E_\beta^2)\right] = \epsilon_0 \sum_\beta \left[\frac{\partial}{\partial x_\beta}(E_\alpha E_\beta) - \frac{1}{2}\frac{\partial}{\partial x_\alpha}(E^2)\right]$$

$$= \epsilon_0 \sum_\beta \left[\frac{\partial}{\partial x_\beta}(E_\alpha E_\beta)\right] - \frac{\epsilon_0}{2}\frac{\partial}{\partial x_\alpha}(E^2) = \epsilon_0 \sum_\beta \frac{\partial}{\partial x_\beta}(E_\alpha E_\beta) - \frac{\epsilon_0}{2}\sum_\beta \frac{\partial}{\partial x_\beta}(E^2)\,\delta_{\alpha\beta}$$

$$= \sum_\beta \epsilon_0 \left[\frac{\partial}{\partial x_\beta}(E_\alpha E_\beta) - \frac{1}{2}\frac{\partial}{\partial x_\beta}(E^2)\,\delta_{\alpha\beta}\right]$$

We know that, $F_{v\alpha} = \sum_\beta \frac{\partial T_{\alpha\beta}}{\partial x_\beta}$.

$\therefore \; T_{\alpha\beta} = \epsilon_0 \left[E_\alpha E_\beta - \frac{1}{2}E^2\delta_{\alpha\beta}\right] = T_{\beta\alpha} \quad [\because \overleftrightarrow{T} \text{ is symmetric}]$.

$$\therefore \overleftrightarrow{T} = \epsilon_0 \begin{bmatrix} \frac{1}{2}(E_x^2 - E_y^2 - E_z^2) & E_x E_y & E_x E_z \\ E_x E_y & \frac{1}{2}(E_y^2 - E_x^2 - E_z^2) & E_y E_z \\ E_x E_z & E_y E_z & \frac{1}{2}(E_z^2 - E_x^2 - E_y^2) \end{bmatrix}$$

This is the Maxwell's electric stress tensor in absence of dielectrics. Diagonal elements represent 'pressures' and off-diagonal elements are 'shears'.

This matrix can be diagonalised, therefore we can get principal values.

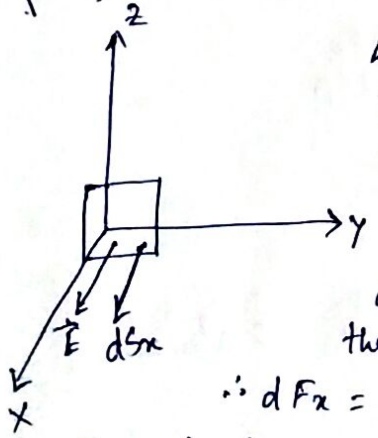
Let $\vec{E} = E\hat{i}$ $\therefore$ $E_x = E$; $E_y = E_z = 0$

$\therefore T_{xx} = \frac{1}{2}E^2$; $T_{xy} = T_{yz} = T_{xz} = 0$; $T_{yy} = -\frac{1}{2}E^2 = T_{zz}$.

$$\therefore \overleftrightarrow{T} = \epsilon_0 \begin{bmatrix} \frac{1}{2}E^2 & 0 & 0 \\ 0 & -\frac{1}{2}E^2 & 0 \\ 0 & 0 & -\frac{1}{2}E^2 \end{bmatrix} = \frac{\epsilon_0}{2}E^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Force element :- $d\vec{F} = \overleftrightarrow{T} d\vec{s}$

$$\begin{pmatrix} dF_x \\ dF_y \\ dF_z \end{pmatrix} = \frac{\epsilon_0 E^2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} ds_x \\ ds_y \\ ds_z \end{pmatrix}$$



Let, $\vec{E} = E\hat{i}$ in yz plane

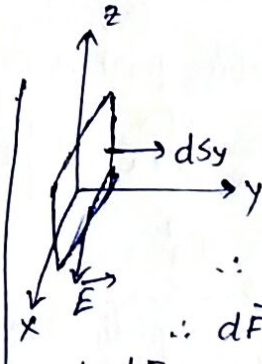
Rectangle of area $d\vec{s} = ds_x\hat{i}$

[$\because$ normal unit areal vector is in a direction (x) $\perp^r$ to the surface plane (yz).

$\therefore dF_x = \frac{\epsilon_0 E^2}{2} ds_x$

Force is in x direction.

Surface will experience a <u>pull</u>. [$\because \vec{E} \parallel d\vec{s}$]



$\vec{E} = E\hat{i}$

Surface is at x-z plane.

$\therefore$ unit vector normal to surface $\hat{j}$.

$\therefore d\vec{s} = ds_y\hat{j}$.

$\therefore d\vec{F} = dF_y\hat{j}$

$\therefore dF_y = -\frac{\epsilon_0 E^2}{2} ds_y$

$-ve$ sign indicates that the surface will experience a <u>push</u>. ($\vec{E} \perp d\vec{s}$).

<u>Stress-tensor</u> in <u>E-M field</u>:- In E-M field, Volume force,

$$\vec{F}_v = \rho\vec{E} + \vec{J}\times\vec{B} \quad \cdots\cdots (1)$$

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \Rightarrow \rho = \epsilon_0(\vec{\nabla}\cdot\vec{E}) \quad \cdots\cdots (2)$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \Rightarrow \vec{J} = \frac{1}{\mu_0}\left[\vec{\nabla}\times\vec{B} - \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right].$$

$$\Rightarrow \vec{J}\times\vec{B} = \frac{1}{\mu_0}\left[(\vec{\nabla}\times\vec{B})\times\vec{B} - \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right] \quad \cdots\cdots (3)$$

[So, what we I have to do that, to evaluate ρ & $\vec{J}$ from Maxwell's equations and put them in (1) and to calculate α-th term of $\vec{F}_v$ ($F_{v\alpha}$) as we did in electrostatic case and then to obtain $T_{\alpha\beta}$ & ultimately $\overleftrightarrow{T}$ in E-M field.]

Now, $\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial\vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{\partial\vec{B}}{\partial t}$; but $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$.

$$= \frac{\partial\vec{E}}{\partial t}\times\vec{B} - \vec{E}\times(\vec{\nabla}\times\vec{E})$$

$$\Rightarrow \frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times(\vec{\nabla}\times\vec{E}).$$

$$\therefore (3) \Rightarrow \vec{J}\times\vec{B} = \frac{1}{\mu_0}\left[-\vec{B}\times(\vec{\nabla}\times\vec{B}) - \mu_0\epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \mu_0\epsilon_0\vec{E}\times(\vec{\nabla}\times\vec{E})\right]$$

$$\therefore (1) \Rightarrow \vec{F}_v = \epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} - \frac{1}{\mu_0}\vec{B}\times(\vec{\nabla}\times\vec{B}) - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \epsilon_0\vec{E}\times(\vec{\nabla}\times\vec{E}) \quad \cdots\cdots (4)$$

$$\vec{B}\times(\vec{\nabla}\times\vec{B}) = \frac{1}{2}\vec{\nabla}B^2 - \vec{B}(\vec{B}\cdot\vec{\nabla}); \quad \vec{E}\times(\vec{\nabla}\times\vec{E}) = \frac{1}{2}\vec{\nabla}E^2 - \vec{E}(\vec{E}\cdot\vec{\nabla})$$

$$(4) \Rightarrow \vec{F}_v = \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} - \frac{1}{2}\vec{\nabla}(E^2) + \vec{E}(\vec{E}\cdot\vec{\nabla})\right] + \frac{1}{\mu_0}\left[-\frac{1}{2}\vec{\nabla}(B^2) + \vec{B}(\vec{B}\cdot\vec{\nabla})\right] - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

[Note that in electric components; there is a term $(\vec{\nabla}\cdot\vec{E})\vec{E}$ which is absent in magnetic components; we can insert that magnetic term $(\vec{\nabla}\cdot\vec{B})\vec{B}$ in $\vec{F}_v$ at any time because we know that $\vec{\nabla}\cdot\vec{B}=0$ always].

$\therefore$ We can write,

$$\vec{F}_v = \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} - \frac{1}{2}\vec{\nabla}E^2\right] - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}B^2\right].$$

α-th component of Electric part :- $\epsilon_0\cancel{\left[E_\alpha\sum_\beta\frac{\partial E_\beta}{\partial x_\beta} + \sum_\beta\right.}$

$$\left[(\vec{\nabla}\cdot\vec{E})\vec{E}\right]_\alpha = E_\alpha\sum_\beta\frac{\partial E_\beta}{\partial x_\beta}; \quad \left[(\vec{E}\cdot\vec{\nabla})\vec{E}\right]_\alpha = \sum_\beta E_\beta\frac{\partial E_\alpha}{\partial x_\beta}$$

$$\left[\frac{1}{2}\vec{\nabla}E^2\right]_\alpha = \frac{1}{2}\frac{\partial}{\partial x_\alpha}\sum_\beta E_\beta^2.$$

α-th component of magnetic part will be same.

$$\epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} - \frac{1}{2}\vec{\nabla}E^2\right]_\alpha = \epsilon_0\left[E_\alpha\sum_\beta\frac{\partial E_\beta}{\partial x_\beta} + \sum_\beta E_\beta\frac{\partial E_\alpha}{\partial x_\beta} - \frac{1}{2}\frac{\partial}{\partial x_\alpha}\sum_\beta E_\beta^2\right]$$

$$= \epsilon_0\left[\sum_\beta\frac{\partial}{\partial x_\beta}(E_\alpha E_\beta) - \frac{1}{2}\sum_\beta\frac{\partial}{\partial x_\beta}(E^2)\delta_{\alpha\beta}\right]$$

$$\therefore F_{v\alpha} = \epsilon_0\left[\sum_\beta\frac{\partial}{\partial x_\beta}(E_\alpha E_\beta) - \frac{1}{2}\sum_\beta\frac{\partial}{\partial x_\beta}(E^2)\delta_{\alpha\beta}\right] - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})_\alpha + \frac{1}{\mu_0}\left[\sum_\beta\frac{\partial}{\partial x_\beta}(B_\alpha B_\beta) - \frac{1}{2}\sum_\beta\frac{\partial}{\partial x_\beta}(B^2)\delta_{\alpha\beta}\right]$$

In E-M field, $T_{\alpha\beta} = \epsilon_0 \left[E_\alpha E_\beta - \frac{1}{2}E^2 \delta_{\alpha\beta}\right] + \frac{1}{\mu_0}\left[B_\alpha B_\beta - \frac{1}{2}B^2 \delta_{\alpha\beta}\right]$

[Note, in electrostatics, $F_{v\alpha} = \sum_\beta \frac{\partial T_{\alpha\beta}}{\partial x_\beta}$].

$T_{xx} = \frac{\epsilon_0}{2}(E_x^2 - E_y^2 - E_z^2) + \frac{1}{2\mu_0}(B_x^2 - B_y^2 - B_z^2)$

$T_{xy} = \epsilon_0 E_x E_y + \frac{1}{\mu_0} B_x B_y$

Define $\vec{E}\times\vec{H} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \vec{N}$ = Poynting vector = Energy flow per unit time per unit area.

$\therefore F_{v\alpha} = \sum_\beta \frac{\partial T_{\alpha\beta}}{\partial x_\beta} - \frac{1}{c^2}\frac{\partial N_\alpha}{\partial t} \Rightarrow \boxed{\vec{F}_v = \vec{\nabla}\cdot\overleftrightarrow{T} - \frac{1}{c^2}\frac{\partial \vec{N}}{\partial t}}$ ⇒ Force per unit volume / Mechanical force density

$\vec{F}_v = \frac{\partial \vec{g}_{mech}}{\partial t}$; $\vec{g}_{mech}$ = Mechanical momentum density.

Now, $\vec{\nabla}\cdot\overleftrightarrow{T} = \vec{F}_v + \frac{1}{c^2}\frac{\partial \vec{N}}{\partial t} = \frac{\partial}{\partial t}(\vec{g}_{mech} + \vec{g}_{em}) = \frac{\partial \vec{g}}{\partial t}$

Conservation Laws [Ref: Panofsky & Phillips, Griffiths]

i) Conservation of momentum :~ $\vec{g}_{em} = \frac{1}{c^2}\vec{N}$ = em momentum density.

$\vec{g} = \vec{g}_{mech} + \vec{g}_{em}$ = Total momentum density.

$\boxed{\vec{\nabla}\cdot\overleftrightarrow{T} = \frac{\partial \vec{g}}{\partial t}}$ ⇒ Conservation of em momentum.

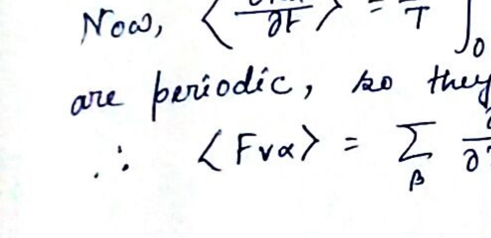
ii) Conservation of energy :~ $\vec{\nabla}\cdot\vec{N} = -\frac{\partial}{\partial t}(U_{mech} + U_{em}) = -\frac{\partial U}{\partial t}$.

$U = U_{mech} + U_{em}$ = Total energy density.

iii) Conservation of charge :~ Continuity equation, $\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \vec{\nabla}\cdot\vec{J} = -\partial\rho/\partial t$.

[Remember: $\vec{N}$ represents energy flux density.
$-\overleftrightarrow{T}$ " momentum " " .
$\vec{N}$ gives information about conservation of em ~~momentum~~ energy.
$\overleftrightarrow{T}$ " " " " momentum.]

Let us evaluate mean / average energy density :~



Consider a black body infinitely extended.

$F_{v\alpha} = \sum_\beta \frac{\partial T_{\alpha\beta}}{\partial x_\beta} - \frac{1}{c^2}\frac{\partial N_\alpha}{\partial t}$.

Taking time average on both sides, we get,

$\langle F_{v\alpha}\rangle = \sum_\beta \frac{\partial}{\partial x_\beta}\langle T_{\alpha\beta}\rangle - \frac{1}{c^2}\left\langle \frac{\partial N_\alpha}{\partial t}\right\rangle$.

Now, $\left\langle \frac{\partial N_\alpha}{\partial t}\right\rangle = \frac{1}{T}\int_0^T \frac{dN_\alpha}{dt}\,dt = \frac{1}{T} N_\alpha \Big|_0^T = 0$, because both $\vec{E}$ & $\vec{B}$ fields are periodic, so they have same value after one time period (T).

$\therefore \langle F_{v\alpha}\rangle = \sum_\beta \frac{\partial}{\partial x_\beta}\langle T_{\alpha\beta}\rangle \Rightarrow \langle F_{vx}\rangle = \frac{\partial\langle T_{xx}\rangle}{\partial x} + \frac{\partial\langle T_{xy}\rangle}{\partial y} + \frac{\partial\langle T_{xz}\rangle}{\partial z}$

16.

$$T_{\alpha\beta} = \epsilon_0 \left(E_\alpha E_\beta - \frac{1}{2} E^2 \delta_{\alpha\beta} \right) + \frac{1}{\mu_0} \left(B_\alpha B_\beta - \frac{1}{2} B^2 \delta_{\alpha\beta} \right)$$

$$T_{xx} = \epsilon_0 \left(E_x^2 - \frac{1}{2} E^2 \right) + \frac{1}{\mu_0} \left(B_x^2 - \frac{1}{2} B^2 \right)$$

$$\langle E_x^2 \rangle = \langle E_y^2 \rangle = \langle E_z^2 \rangle = \frac{1}{3} \langle E^2 \rangle$$

$$\langle E^2 \rangle = \langle E_x^2 \rangle + \langle E_y^2 \rangle + \langle E_z^2 \rangle = 3 \langle E_x^2 \rangle .$$

$$\therefore \langle T_{xx} \rangle = \epsilon_0 \left[\frac{1}{3} \langle E^2 \rangle - \frac{1}{2} \langle E^2 \rangle \right] + \frac{1}{\mu_0} \left[\frac{1}{3} \langle B^2 \rangle - \frac{1}{2} \langle B^2 \rangle \right] .$$

$$= - \frac{1}{6} \left[\epsilon_0 \langle E^2 \rangle + \frac{1}{\mu_0} \langle B^2 \rangle \right] = - \frac{1}{3} \langle U \rangle .$$

$\langle U \rangle = \frac{1}{2} \left(\epsilon_0 \langle E^2 \rangle + \frac{1}{\mu_0} \langle B^2 \rangle \right) =$ Mean/ Average energy density.

Similarly, $\langle T_{xy} \rangle = \epsilon_0 \langle E_x E_y \rangle + \frac{1}{\mu_0} \langle B_x B_y \rangle$

As the radiation is unpolarised, these components of $\vec{E}$ are uncorrelated. So, $\langle T_{xy} \rangle = \langle T_{xz} \rangle = 0$.