# VIVEKANANDA COLLEGE THAKURPUKUR KOLKATA-700063

NAAC ACCREDITED 'A' GRADE

Topic: Radiation from time-dependent sources of charges and currents

Course Title: Classical Electrodynamics

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Inhomogeneous wave equations and their solutions :- [Ref: Jackson, Panofsky & Phillips]

In previous chapter, we have got inhomogeneous wave equation in static case, $\nabla^2\phi(\vec{r},t) = -\rho(\vec{r},t)/\epsilon_0$ ......① and its solution, $\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int_{v'} \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}dv'$ ...②

In presence of time-dependent term, Maxwell's inhomogeneous ~~wave~~ equations (3 & 4, i.e; $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ & $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$) takes forms,

$\Box^2\phi + \frac{\partial L}{\partial t} = -\rho/\epsilon_0$ & $\Box^2\vec{A} - \vec{\nabla}L = -\mu_0\vec{J}$; [$\rho$ & $\vec{J}$ are source fields.]

Using Lorentz gauge, $L=0$, those inhomogeneous equations reduced to,

$\Box^2\phi = -\rho/\epsilon_0$ .....③ & $\Box^2\vec{A} = -\mu_0\vec{J}$ ....④

where, $\Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

∴ Wave equations ③ & ④ have the basic structure, $\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -g(\vec{r},t)$

$\Rightarrow \Box^2\psi(\vec{r},t) = -g(\vec{r},t)$ ....⑤ [$\psi$ is the general form of potential $\phi$ & $\vec{A}$ & $g$ is the general source]

where $g(\vec{r},t)$ is a source function & it can be analyzed by the Fourier integral, $g(\vec{r},t) = \int_{-\infty}^{\infty} g_\omega(\vec{r})\, e^{-i\omega t}\, d\omega$ .....⑥ [$g_\omega(\vec{r}) \equiv g(\vec{r},\omega)$]

which has the Fourier inversion, $g_\omega(\vec{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\vec{r},t)\, e^{i\omega t}\, dt$ ...⑦

Similarly, we may analyze the general potential $\psi(\vec{r},t)$ into Fourier components, $\psi(\vec{r},t) = \int_{-\infty}^{\infty}\psi_\omega(\vec{r})\, e^{-i\omega t}\, d\omega$ ....⑧

with a corresponding inverse relation, $\psi_\omega(\vec{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(\vec{r},t)\, e^{i\omega t} dt$ ...⑨

Substituting ⑥ & ⑧ into equation ⑤, we get, $\nabla^2\psi_\omega + \frac{\omega^2}{c^2}\psi_\omega = -g_\omega$ ......⑩ which is similar to Poisson's equation.

To get the solution of equation ⑧, superpose unit point solutions corresponding to a source at the point $\vec{r}'$ given by, $g_\omega(\vec{r}) = \delta(\vec{r}-\vec{r}')$, where, $\delta(\vec{r}-\vec{r}')$ is the 'Dirac δ-function'. $R = |\vec{R}| = \vec{r}-\vec{r}'$

Each unit source potential satisfies the equation,

$$\nabla^2 G(\vec{r},\vec{r}') + \frac{\omega^2}{c^2}G(\vec{r},\vec{r}') = -\delta(\vec{r}-\vec{r}') \quad \text{....⑪}$$

with 'G' known as Green function is a function of both $\vec{r}$ & $\vec{r}'$. $\omega/c = K$ = wave number.

The partial solution corresponding to the frequency ω of the total source is then given by the superposition

$$\psi_\omega(\vec{r}) = \int g_\omega(\vec{r}')\, G(\vec{r},\vec{r}')\, dv' \quad \text{....⑫}$$

To get the solution for potential, first ~~to~~ we have to evaluate G. using equation ⑪ & assuming the equation spherically symmetric in $R$ ($R = |\vec{r}-\vec{r}'|$), that is, it depends only on R. $\nabla^2$ in spherical coordinate system is, $\nabla^2 \equiv \frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{d}{dR}\right)$ [∵ only R dependence due to spherically symmetric]

At all points other than $R=0$, equation (11) reduces to,

$$\nabla^2 G_\omega(\vec{r},\vec{r}\,') + k^2 G_\omega(\vec{r},\vec{r}\,') = 0 \quad \cdots\cdots (13)$$

Let us determine first $\nabla^2 G_\omega = \frac{1}{R^2}\frac{d}{dR}\left(R^2 \frac{dG_\omega}{dR}\right)$.

$$\frac{d}{dR}\left(R^2\frac{dG}{dR}\right) = R^2\frac{d^2G}{dR^2} + 2R\frac{dG}{dR}.$$

$$\therefore \frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dG}{dR}\right) = \frac{d^2G}{dR^2} + \frac{2}{R}\frac{dG}{dR} = \nabla^2 G \quad \cdots\cdots (i)$$

Now, $\frac{d}{dR}(RG) = G + R\frac{dG}{dR}$.

$$\therefore \frac{d^2}{dR^2}(RG) = \frac{dG}{dR} + \frac{d}{dR}\left(R\frac{dG}{dR}\right) = \frac{dG}{dR} + R\frac{d^2G}{dR^2} + \frac{dG}{dR} = R\frac{d^2G}{dR^2} + 2\frac{dG}{dR}$$

$$\therefore \frac{1}{R}\frac{d^2}{dR^2}(RG) = \frac{d^2G}{dR^2} + \frac{2}{R}\frac{dG}{dR} = \nabla^2 G \quad [\text{See } (i)].$$

$$\therefore (13) \Rightarrow \frac{1}{R}\frac{d^2}{dR^2}(RG_\omega) + k^2 G_\omega = 0$$

[~~Remember,~~ Remember, $R = |\vec{r}-\vec{r}\,'|$]

$$\Rightarrow \frac{d^2}{dR^2}(RG_\omega) + k^2(RG_\omega) = 0$$

$$\Rightarrow RG_\omega = A e^{\pm ikR} \Rightarrow G_\omega = \frac{A}{R}e^{\pm ikR}.$$

$$\therefore G_\omega(\vec{r},\vec{r}\,') = \frac{A}{|\vec{r}-\vec{r}\,'|} e^{\pm ik|\vec{r}-\vec{r}\,'|} \quad \cdots\cdots (14)$$

We have to evaluate constant $A$. Suppose, $R \to 0$.

For $R \to 0$, $G_\omega \to \frac{A}{R}$.

(13) for $R \to 0 \Rightarrow \nabla^2 G_\omega + k^2 G_\omega = -\delta(\vec{R}) \Rightarrow \int(\nabla^2 G_\omega + k^2 G_\omega)dv' = -\int \delta(\vec{R})dv'$ $\cdots\cdots$ (15)

Now, $R \to 0$, $G_\omega \to A/R$.

$$\vec{\nabla}(G_\omega) = \vec{\nabla}(A/R) = -\frac{A}{R^2}\hat{R}$$

$$\int \nabla^2(A/R)\,dv' = \int \vec{\nabla}\cdot\left(-\frac{A\hat{R}}{R^2}\right)dv' = -\int \frac{A\hat{R}}{R^2}\cdot d\vec{s} = -A\cdot\frac{\hat{R}}{R^2}\cdot 4\pi R^2\cdot\hat{R}$$

$$= -4\pi A. \qquad [\because d\vec{s} = 4\pi R^2\cdot\hat{R}]$$

$$\int k^2 G_\omega dv' = \int (k^2 A/R)dv' = \int \frac{k^2 A}{R}\cdot 4\pi R^2\, dR = k^2 A\cdot 4\pi \int R\,dR = 0 \text{ for } R \to 0.$$

$$\int -\delta(\vec{R})dv' = -1.$$

$$(15) \Rightarrow -4\pi A = -1 \Rightarrow A = 1/4\pi.$$

$\therefore$ Green function becomes, (14) $\Rightarrow G_\omega(\vec{r},\vec{r}\,') = \frac{1}{4\pi|\vec{r}-\vec{r}\,'|}e^{\pm ik|\vec{r}-\vec{r}\,'|}$ $\cdots\cdots$ (16)

(12) becomes, $\psi(\vec{r}) = \int g_\omega(\vec{r}\,')\cdot\frac{1}{4\pi|\vec{r}-\vec{r}\,'|}e^{\pm ik|\vec{r}-\vec{r}\,'|}dv'$ $\cdots\cdots$ (17)

[ Note: If you compare the solutions of $\nabla^2\phi$ & $\Box^2\psi$, ~~we can see~~ (equations (2) & (17), we can see, that (17) has an extra term $e^{\pm ikR}$ & generally this term comes from $\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$ component. So, $\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$ has a solution $e^{\pm ikR}$. ]

∴ Time dependent general potential $\psi(\vec{r},t)$ is given by [Using (8) & (17)]

$$\psi(\vec{r},t) = \int \psi_\omega(\vec{r})\, e^{-i\omega t}\, d\omega = \frac{1}{4\pi}\iint \frac{g_\omega(\vec{r}')\, e^{-i(\omega t \pm KR)}}{R}\, d\omega\, dv' \quad \cdots (18)$$

Now, $\omega t \pm KR = \omega(t \pm KR/\omega) = \omega(t \pm R/c)$

We can introduce a new time $t'(\vec{r},\vec{r}') = t \pm R/c$.

We have discussed about advanced and retarded times in the 1st chapter. We are interested only in retarded time $t' = t - R/c$.

$$\therefore (18) \Rightarrow \psi(\vec{r},t) = \frac{1}{4\pi}\iint \frac{g_\omega(\vec{r}')\, e^{-i\omega t'}}{R}\, d\omega\, dv' = \frac{1}{4\pi}\int \frac{g(\vec{r}',t')}{R}\, dv' \quad \cdots (19)$$

$$\left[\because g(\vec{r},t) = \int g_\omega(\vec{r})\, e^{-i\omega t}\, d\omega\right]$$

This is the solution of general potential at retarded time or we can say solution of general retarded potential $\psi(\vec{r},t)$.

[ Please check retarded potential at chapter 1 ].

$$(18) \Rightarrow \psi(\vec{r},t) = \int \psi_\omega(\vec{r})\, e^{-i\omega t}\, d\omega = \frac{1}{4\pi}\iint \frac{g_\omega(\vec{r}')\, e^{-i\omega t}\, e^{\pm iKR}}{R}\, d\omega\, dv'$$

$$= \frac{1}{4\pi}\iint \frac{g(\vec{r}',t)}{R}\, e^{\pm iKR}\, dv' \quad \cdots (20)$$

This is also a solution of general retarded potential $\psi(\vec{r},t)$.

## Radiation [Ref: Griffiths, Jackson]

All electromagnetic waves or fields have sources consisting of electric charges. But a charge at rest or a steady current can not generate e-m waves. It takes accelerating charges and changing currents. We know that, e-m waves propagate out to infinity in vacuum, carrying energy with them ; the signature of radiation is this irreversible flow of energy away from the source. Remember static sources do not radiate.

The study of radiation involves picking out the parts of $\vec{E}$ & $\vec{B}$ that go like $1/R$ at large distances from the source.

Okay, we now back to our present calculations. We are basically dealing with two potentials $\vec{A}(\vec{r})$ & $\phi(\vec{r})$.

$$(20) \Rightarrow \vec{A}_\omega(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}_\omega(\vec{r}')\, e^{iKR}}{R}\, dv' \quad \cdots (21)$$

$$\& \quad \phi_\omega(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_\omega(\vec{r}')\, e^{iKR}}{R}\, dv' \quad \cdots (22)$$

$$\text{with,} \quad \vec{A}(\vec{r},t) = \int_{-\infty}^{\infty} \vec{A}_\omega(\vec{r})\, e^{-i\omega t}\, d\omega \quad \cdots (23)$$

$$\& \quad \phi(\vec{r},t) = \int_{-\infty}^{\infty} \phi_\omega(\vec{r})\, e^{-i\omega t}\, d\omega \quad \cdots (24)$$

[Remember : $\rho(\vec{r};t) = \int_{-\infty}^{\infty} \rho_\omega(\vec{r}')\, e^{-i\omega t}\, d\omega \quad \cdots\cdots (25)$

$\vec{J}(\vec{r};t) = \int_{-\infty}^{\infty} \vec{J}_\omega(\vec{r}')\, e^{-i\omega t}\, d\omega \quad \cdots\cdots (26)$

$t' = t - R/c$ = retarded time

$(25) \Rightarrow \rho(\vec{r};t') = \int_{-\infty}^{\infty} \rho_\omega(\vec{r}')\, e^{-i\omega t'}\, d\omega = \int_{-\infty}^{\infty} \rho_\omega(\vec{r}')\, e^{-i\omega t}\, e^{i\omega R/c}\, d\omega$

$= \int_{-\infty}^{\infty} \rho_\omega(\vec{r}')\, e^{-i\omega t}\, e^{+ikR}\, d\omega \quad \cdots (27)$

$\vec{J}(\vec{r};t') = \int_{-\infty}^{\infty} \vec{J}_\omega(\vec{r}')\, e^{-i\omega t'}\, d\omega = \int_{-\infty}^{\infty} \vec{J}_\omega(\vec{r}')\, e^{-i\omega t}\, e^{ikR}\, d\omega \quad \cdots\cdots (28)$ ]

Now, we will evaluate $\vec{B}(\vec{r},t)$ & $\vec{E}(\vec{r},t)$ in terms of $\rho(\vec{r};t')$ & $\vec{J}(\vec{r};t')$.

<u>Jefimenko's equations</u> :-

We know that, $\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \vec{B}_\omega(\vec{r}) = \vec{\nabla}\times\vec{A}_\omega(\vec{r})$.

& $\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$.

$(21) \Rightarrow \vec{A}_\omega(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}_\omega(\vec{r}')\, e^{ikR}}{R}\, dv'$

$\Rightarrow \vec{\nabla}\times\vec{A}_\omega(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left(\frac{\vec{J}_\omega(\vec{r}')\, e^{ikR}}{R}\right) dv'$

$= \frac{\mu_0}{4\pi}\int \left[\frac{e^{ikR}}{R}\left\{\vec{\nabla}\times\vec{J}_\omega(\vec{r}')\right\} + \left\{\vec{\nabla}\left(\frac{e^{ikR}}{R}\right)\times\vec{J}_\omega(\vec{r}')\right\}\right] dv'$

First term should be zero. $\vec{\nabla}\times\vec{J}_\omega(\vec{r}') = 0$, because $\vec{\nabla}$ is in $\vec{r}(x,y,z)$ i.e; other co-ordinate system.

$\left[\vec{\nabla}\left(\frac{e^{ikR}}{R}\right) = \frac{1}{R}\cdot ik\, e^{ikR}\hat{R} - \frac{e^{ikR}}{R^2}\hat{R} = \frac{ik\, e^{ikR}\vec{R}}{R^2} - \frac{e^{ikR}\vec{R}}{R^3}\right]$

$\therefore \vec{B}_\omega(\vec{r}) = \vec{\nabla}\times\vec{A}_\omega(\vec{r}) = \frac{\mu_0}{4\pi}\int -\left[\vec{J}_\omega(\vec{r}')\times\vec{\nabla}\left(\frac{e^{ikR}}{R}\right)\right] dv'$

$= -\frac{\mu_0}{4\pi}\int \left[\frac{\vec{J}_\omega(\vec{r}')\times\vec{R}}{R^2}\left(ik\, e^{ikR}\right) - \frac{\vec{J}_\omega(\vec{r}')\times\vec{R}}{R^3}\, e^{ikR}\right] dv'$

$= \frac{\mu_0}{4\pi}\int \frac{\vec{J}_\omega(\vec{r}')\times\vec{R}}{R^3}\, e^{ikR}\, dv' - \frac{\mu_0}{4\pi}\int \frac{\vec{J}_\omega(\vec{r}')\times\vec{R}}{R^2}\left(ik\, e^{ikR}\right) dv'$

$\vec{B}(\vec{r};t) = \int \vec{B}_\omega(\vec{r})\, e^{-i\omega t}\, d\omega \; \cancel{= \frac{\mu_0}{4\pi}\left[\int \frac{\{\vec{J}_\omega(\vec{r}')\, e^{-i\omega t}\, e^{ikR}\times\vec{R}\}}{R^3}\, dv' - \int \frac{\{\vec{J}_\omega(\vec{r}')\, e^{-i\omega t}\, e^{ikR}\times\vec{R}\}\, ik}{R^2}\, dv'\right]}$

$= \frac{\mu_0}{4\pi}\left[\iint \frac{\vec{J}_\omega(\vec{r}')\, e^{-i\omega t}\, e^{ikR}\times\vec{R}}{R^3}\, d\omega\, dv' - \iint \frac{\vec{J}_\omega(\vec{r}')\, e^{-i\omega t}\, e^{ikR}\times\vec{R}}{R^2}\,(ik)\, d\omega\, dv'\right]$

$= \frac{\mu_0}{4\pi}\left[\int \frac{\vec{J}(\vec{r}';t')\times\vec{R}}{R^3}\, dv' - \int \frac{ik\,\vec{J}(\vec{r};t')\times\vec{R}}{R^2}\, dv'\right]$ [Using (28) ] $\cdots (29)$

5.

Now, $\dot{\vec{J}}(\vec{r}',t') = \frac{\partial}{\partial t'} \vec{J}(\vec{r}',t') = \frac{\partial}{\partial t'}\int \vec{J}_\omega(\vec{r}')\, e^{-i\omega t'} d\omega = (-i\omega)\vec{J}(\vec{r}',t')$

$\therefore$ (29) $\Rightarrow \vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\left[\int \frac{\vec{J}(\vec{r}',t')\times\vec{R}}{R^3}\, dv' + \int \frac{\dot{\vec{J}}(\vec{r}',t')\times\vec{R}}{cR^2}\, dv'\right]$

$$\Rightarrow \vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{V'}\left[\frac{\vec{J}(\vec{r}',t')}{R^2} + \frac{\dot{\vec{J}}(\vec{r}',t')}{cR}\right]\times\hat{R}\, dv' \quad \ldots\ldots (i)$$

Equation (i) is the time-dependent generalization of the Biot-Savart law. You can see that the first term is simply Biot-Savart law in static case & second term is the time-dependent term.

$\vec{E}_\omega(\vec{r}) = -\vec{\nabla}\phi_\omega(\vec{r}) - \frac{\partial \vec{A}_\omega(\vec{r})}{\partial t}$

$\vec{E}(\vec{r},t) = -\vec{\nabla}\phi(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}$

$\phi(\vec{r},t) = \int \phi_\omega(\vec{r})\, e^{-i\omega t} d\omega = \iint \frac{1}{4\pi\epsilon_0}\, \frac{\rho_\omega(\vec{r}')\, e^{ikR} e^{-i\omega t}}{R}\, d\omega\, dv'$ [Using (22) & (24)]

$= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t')}{R}\, dv'$

$\therefore \vec{\nabla}\phi = \frac{1}{4\pi\epsilon_0}\int \vec{\nabla}(\rho/R)\, dv'$

Now, $\vec{\nabla}(\rho/R) = \frac{\vec{\nabla}\rho}{R} + \rho\,\vec{\nabla}\left(\frac{1}{R}\right)$

$\vec{\nabla}\rho = \frac{\partial\rho}{\partial t'}\vec{\nabla}t' = \dot{\rho}(\vec{r}',t')\,\vec{\nabla}t'$

$t' = t - R/c \Rightarrow \vec{\nabla}t' = -\frac{1}{c}\vec{\nabla}R = -\frac{1}{c}\hat{R}$

& $\vec{\nabla}\frac{1}{R} = -\frac{1}{R^2}\hat{R}$.

$\therefore \vec{\nabla}(\rho/R) = \frac{\vec{\nabla}\rho}{R} + \rho\,\vec{\nabla}\left(\frac{1}{R}\right) = -\frac{\dot{\rho}(\vec{r}',t')}{cR}\hat{R} - \frac{\rho}{R^2}\hat{R}$

$\therefore \vec{\nabla}\phi(\vec{r},t) = \frac{-1}{4\pi\epsilon_0}\int\left[\frac{\dot{\rho}(\vec{r}',t')}{cR}\hat{R} + \frac{\rho(\vec{r}',t')}{R^2}\hat{R}\right] dv'$

Similarly, $\vec{A}(\vec{r},t) = \int \vec{A}_\omega(\vec{r})\, e^{-i\omega t} d\omega = \frac{\mu_0}{4\pi}\iint \frac{\vec{J}_\omega(\vec{r}')\, e^{ikR} e^{-i\omega t}}{R}\, d\omega\, dv'$ [Using (21) & (23)]

$= \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t')}{R}\, dv'$

$\frac{\partial \vec{A}(\vec{r},t)}{\partial t} = \frac{\mu_0}{4\pi}\int \frac{\dot{\vec{J}}(\vec{r}',t')}{R}\, dv'$

$\therefore \vec{E}(\vec{r},t) = -\vec{\nabla}\phi(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}$

$= \frac{1}{4\pi\epsilon_0}\int\left[\frac{\dot{\rho}(\vec{r}',t')}{cR}\hat{R} + \frac{\rho(\vec{r}',t')}{R^2}\hat{R}\right] dv' - \frac{\mu_0\epsilon_0}{4\pi\epsilon_0}\int \frac{\dot{\vec{J}}(\vec{r}',t')}{R}\, dv'$

$$= \frac{1}{4\pi\epsilon_0}\int_{V'}\left[\frac{\rho(\vec{r}',t')}{R^2}\hat{R} + \frac{\dot{\rho}(\vec{r}',t')}{cR}\hat{R} - \frac{\dot{\vec{J}}(\vec{r}',t')}{c^2R}\right] dv' \quad \ldots\ldots (ii)$$

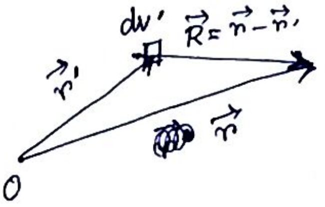
6.

This is the time dependent generalization of Coulomb's law. Note that the first term is simply Coulomb's law in static case.

Equations (i) and (ii) are known as 'Jefimenko's ~~law~~ equations'.

**Radiation components of $\vec{E}$ and $\vec{B}$ :-**

Let us consider the field point $\vec{r}$ is far away from the source point $\vec{r}' \Rightarrow |\vec{r}'| << |\vec{r}|$




$$|\vec{R}| = R = |\vec{r} - \vec{r}'| = (r^2 + r'^2 - 2\vec{r}\cdot\vec{r}')^{1/2}$$

Neglecting $r'^2$ ($\because r'$ is very small) we get,

$$R = (r^2 - 2\vec{r}\cdot\vec{r}')^{1/2} = r\left(1 - 2\frac{\vec{r}\cdot\vec{r}'}{r^2}\right)^{1/2} \simeq r\left(1 - \frac{\hat{r}\cdot\vec{r}'}{r}\right) \quad \cdots\cdots (30)$$

$$\frac{1}{R} = \frac{1}{r}\left(1 - 2\frac{\hat{r}\cdot\vec{r}'}{r}\right)^{-1/2} \simeq \frac{1}{r}\left(1 + \frac{\hat{r}\cdot\vec{r}'}{r}\right) \quad \cdots\cdots (31)$$

$$\rho(\vec{r}', t') = \rho(\vec{r}', t - R/c) \simeq \rho\left(\vec{r}', t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}\right)$$ [Substituting the value of R in (30)]

Let, $t - r/c \equiv t_0$ = retarded time at the origin.

$$\therefore \rho(\vec{r}', t') = \rho\left(\vec{r}', t_0 + \frac{\hat{r}\cdot\vec{r}'}{c}\right) \quad \cdots\cdots (32)$$

Expanding $\rho$ as a Tayer series in 't' about the retarded time at the origin '$t_0$', we have,

$$\rho(\vec{r}'; t') \simeq \rho\left(\vec{r}'; t_0 + \frac{\hat{r}\cdot\vec{r}'}{c}\right) \simeq \rho(\vec{r}'; t_0) + \dot{\rho}(\vec{r}'; t_0)\left(\frac{\hat{r}\cdot\vec{r}'}{c}\right) + \frac{1}{2!}\ddot{\rho}\left(\frac{\hat{r}\cdot\vec{r}'}{c}\right)^2 + \frac{1}{3!}\dddot{\rho}\left(\frac{\hat{r}\cdot\vec{r}'}{c}\right)^3 + \cdots \quad \cdots\cdots (33)$$

[Note that zeroth-order terms of $r'$ are monopole term, first-order terms of $r'$ are dipole term, second-order terms of $r'$ are quadrupole terms and so on.]

Here, we consider upto 1st-order terms of $r'$. Equation (33) reduces to,

$$\therefore \rho(\vec{r}', t') \simeq \rho(\vec{r}'; t_0) + \dot{\rho}(\vec{r}'; t_0)\left(\frac{\hat{r}\cdot\vec{r}'}{c}\right) \quad \cdots\cdots (34)$$

$$\frac{\rho(\vec{r}', t')}{R} \simeq \left\{\rho + \dot{\rho}\left(\frac{\hat{r}\cdot\vec{r}'}{c}\right)\right\}\frac{1}{r}\left(1 + \frac{\hat{r}\cdot\vec{r}'}{r}\right) \quad \cdots$$ [Using (31) & (34)]

$$= \frac{\rho}{r} + \frac{\dot{\rho}(\hat{r}\cdot\vec{r}')}{rc} + \frac{\rho(\hat{r}\cdot\vec{r}')}{r^2} \quad \cdots\cdots (35)$$ [Neglecting $r'^2$ term].

$$\therefore \phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho(\vec{r}', t')}{R}\, dv'$$

$$= \frac{1}{4\pi\epsilon_0 r}\left[\int \rho(\vec{r}'; t_0)\, dv' + \frac{\hat{r}}{r}\cdot\int \vec{r}'\rho(\vec{r}'; t_0)\, dv' + \frac{\hat{r}}{c}\cdot\int \frac{d}{dt}\{\vec{r}'\rho(\vec{r}'; t_0)\}\, dv'\right] \quad \cdots (36)$$

The first integral is simply the total charge Q at time $t_0$. Because charge is conserved, $Q(t_0)$ is actually independent of time. The other two integrals represent the electric dipole moment at time $t_0$. The first term is basically electric monopole moment at time $t_0$.

$$(36) \Rightarrow \phi(\vec{r}, t) \simeq \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{r}\cdot\vec{p}(\vec{r}; t_0)}{r^2} + \frac{\hat{r}\cdot\dot{\vec{p}}(\vec{r}; t_0)}{rc}\right] \quad \cdots\cdots (37)$$

First two terms of equation (37) are monopole & dipole contributions to the multipole expansion for $\phi$ respectively. The third term is time-dependent term.

Like $\rho(\vec{r}';t')$ we can expand $\vec{J}(\vec{r}';t')$ in the same way & we have,

$$\vec{J}(\vec{r}';t') \simeq \vec{J}(\vec{r}',t_0) + \dot{\vec{J}}(\vec{r}';t_0)\left(\frac{\hat{r}\cdot\vec{r}'}{c}\right) + \cdots \quad (38) \quad [\text{Same like } (34)]$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}';t')}{R}\, dv'$$

Let us calculate, $\int \vec{\nabla}'\cdot(x'\vec{J})\, dv'$.

$$\vec{\nabla}'\cdot(x'\vec{J}) = \vec{\nabla}'x'\cdot\vec{J} + x'\,\vec{\nabla}'\cdot\vec{J} = J_{x'} + x'(\vec{\nabla}'\cdot\vec{J}).$$

$$\left[\int \vec{J}(\vec{r}',t')\, dv'\right]_{x'} = \int J_{x'}\, dv' = \int \vec{\nabla}'\cdot(x'\vec{J})\, dv' - \int x'(\vec{\nabla}'\cdot\vec{J})\, dv' \quad \cdots (39)$$

$\int \vec{\nabla}'\cdot(x'\vec{J})dv'$ can be converted into surface term and for arbitrary large surface, the integral would be zero.

$$(39) \Rightarrow \int J_{x'}\, dv' = -\int x'(\vec{\nabla}'\cdot\vec{J})\, dv' \quad \cdots (40)$$

Equation of continuity, $\vec{\nabla}'\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0 \Rightarrow \vec{\nabla}'\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$

$$\Rightarrow \int x'(\vec{\nabla}'\cdot\vec{J})dv' = -\int x'\frac{\partial\rho}{\partial t}\, dv' \Rightarrow -\int J_{x'}\, dv' = -\int x'\frac{\partial\rho}{\partial t}\, dv' \quad \cdots (41)$$

[Using (40)]

$$\rho(\vec{r}';t) = \int \rho_\omega(\vec{r}')\, e^{-i\omega t}\, d\omega.$$

$$\frac{\partial\rho}{\partial t} = (-i\omega)\rho$$

$(41) \Rightarrow \int J_{x'}\, dv' = \int (-i\omega)\rho x'\, dv' = (-i\omega)p_{x'}$ [ $p_{x'}$ = $x'$th component of electric dipole moment $\vec{p}(\vec{r};t)$].

$$\Rightarrow \int \vec{J}(\vec{r}';t')\, dv' = (-i\omega)\vec{p}(\vec{r};t) = \dot{\vec{p}}(\vec{r};t) \quad \cdots (42)$$

$\left[\vec{p}(\vec{r};t) = \vec{p}(\vec{r}')\, e^{-i\omega t} \quad \therefore \dot{\vec{p}}(\vec{r};t) = (-i\omega)\vec{p}(\vec{r};t)\right]$

$$\therefore \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}';t')}{R}\, dv'.$$

$$\simeq \frac{\mu_0}{4\pi}\,\frac{\dot{\vec{p}}(\vec{r};t_0)}{r} \quad \left[\text{Taking } \frac{1}{R} \approx \frac{1}{r} \text{ \& } \dot{\vec{p}}(\vec{r};t') \simeq \dot{\vec{p}}(\vec{r},t_0)\right] \quad \cdots (43)$$

It can be noted that $\dot{\vec{p}}(\vec{r};t_0)$ is already 1st-order term of $r'$, so we have neglected higher order terms of $r'$.

We have got first order terms of $r'$ in $\phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$. Next we must calculate the fields $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$. We are now interested in the radiation zone, so we keep only those terms in fields, which go like $1/r$.

For instance, the Coulomb's field in electrostatics, $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$ and it is coming from the first term of equation (37) and it does not contribute to the em radiation (as it goes like $1/r^2$). Similarly, the second term of (37) goes like $1/r^2$. So, gradient of second term should go like,

8. $\vec{\nabla}\left(\frac{\hat{r}\cdot\vec{p}}{r^2}\right) \propto \frac{1}{r^2}$ & $\frac{1}{r^3}$. So, it does not contribute to the em radiation.

$\therefore$ (37) $\Rightarrow$ $\vec{\nabla}\phi \simeq \vec{\nabla}\left[\frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\dot{\vec{p}}(t_0)}{rc}\right] \simeq \frac{1}{4\pi\epsilon_0}\left[\frac{\hat{r}\cdot\ddot{\vec{p}}(t_0)}{rc}\right]\vec{\nabla}t_0$ $\left[\because \vec{\nabla}\dot{p}(t_0)=\ddot{p}(t_0)\vec{\nabla}t_0\right.$ consider only $1/r$ terms$\left.\right]$

$= -\frac{1}{4\pi\epsilon_0 c^2}\frac{[\hat{r}\cdot\ddot{\vec{p}}(t_0)]}{r}\hat{r}$ ..... (44)

$\vec{\nabla}t_0 = \vec{\nabla}(t-r/c)$
$= -\frac{1}{c}\hat{r}$.

(43) $\Rightarrow$ $\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{\dot{\vec{p}}(t_0)}{r}$

$\frac{\partial\vec{A}(\vec{r},t)}{\partial t} = \frac{\mu_0}{4\pi}\frac{\ddot{\vec{p}}(t_0)}{r}$ ...... (45)

$\vec{\nabla}\times\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\left[\vec{\nabla}\times\frac{\dot{\vec{p}}(t_0)}{r}\right] = \frac{\mu_0}{4\pi r}\vec{\nabla}\times\dot{\vec{p}}(t_0)$ $\left[\text{Neglecting } \vec{\nabla}\frac{1}{r}\times\dot{\vec{p}}(t_0) \text{ as } \vec{\nabla}\frac{1}{r} \text{ goes like } 1/r^2\right]$.

$= \frac{\mu_0}{4\pi r}\left[\vec{\nabla}t_0\times\ddot{\vec{p}}(t_0)\right] = -\frac{\mu_0}{4\pi rc}\left[\hat{r}\times\ddot{\vec{p}}(t_0)\right]$ .... (46)

$\vec{E}_{rad} = \vec{E}(\vec{r},t)\big|_{rad} = -\vec{\nabla}\phi(\vec{r},t) - \frac{\partial\vec{A}(\vec{r},t)}{\partial t}$

$\simeq +\frac{1}{4\pi\epsilon_0 c^2}\frac{\hat{r}\cdot\ddot{\vec{p}}(t_0)}{r}\hat{r} - \frac{\mu_0}{4\pi}\frac{\ddot{\vec{p}}(t_0)}{r}$ [Using (44) & (45)].

$\simeq \frac{\mu_0}{4\pi r}\left[\{\hat{r}\cdot\ddot{\vec{p}}(t_0)\}\hat{r} - \ddot{\vec{p}}(t_0)\right] = \frac{\mu_0}{4\pi r}\left[\hat{r}\times(\hat{r}\times\ddot{\vec{p}})\right]$ ........ (i)

[Note that $\vec{p}(t_0) = \vec{p}(\vec{r}',t_0) = \vec{p}$]

$\vec{B}_{rad} = \vec{B}(\vec{r},t)\big|_{rad} = -\frac{\mu_0}{4\pi rc}\left[\hat{r}\times\ddot{\vec{p}}\right]$ ....... (ii) [Using (46)

From (i) & (ii) we have, $\vec{E}_{rad} = -c(\hat{r}\times\vec{B}_{rad}) = c(\vec{B}_{rad}\times\hat{r})$ ..... (47)

$\therefore$ $\vec{E}_{rad}$, $\vec{B}_{rad}$ and $\hat{r}$ are orthogonal systems.

Poynting vector $\vec{N} = \frac{1}{\mu_0}(\vec{E}_{rad}\times\vec{B}_{rad}) = \frac{c}{\mu_0}(\vec{B}_{rad}\times\hat{r})\times\vec{B}_{rad}$ [Using (47)]

$= \frac{c}{\mu_0}B_{rad}^2\hat{r}$ ..... (48)

If we use spherical polar co-ordinates, with the z-axis in the direction of $\ddot{\vec{p}}(\vec{r};t_0)$, then,

$\vec{E}_{rad}(r,\theta,t) \simeq \frac{\mu_0\ddot{p}(t_0)}{4\pi}\left(\frac{\sin\theta}{r}\right)\hat{\theta}$ .... (49)

$\vec{B}_{rad}(r,\theta,t) \simeq \frac{\mu_0\ddot{p}(t_0)}{4\pi c}\left(\frac{\sin\theta}{r}\right)\hat{\phi}$ ..... (50)

$\therefore \vec{N} \simeq \frac{1}{\mu_0}(\vec{E}_{rad}\times\vec{B}_{rad}) = \frac{\mu_0\ddot{p}^2(t_0)}{16\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$ ..... (51)

Total radiated power $P \simeq \int\vec{N}\cdot d\vec{s} = \frac{\mu_0\ddot{p}^2}{16\pi^2 c}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{\sin^2\theta}{r^2}\sin\theta\, r^2\,d\theta\,d\phi$

$= \frac{\mu_0\ddot{p}^2}{16\pi^2 c}\times\frac{4}{3}\times 2\pi = \frac{\mu_0\ddot{p}^2}{6\pi c}$ .... (52)

Notice that $\vec{E}_{rad}$ and $\vec{B}_{rad}$ are mutually perpendicular, transverse to the direction of propagation ($\hat{r}$) and $E_{rad}/B_{rad} = c$ always for radiation fields. $p(t_0) = q\, r'(t_0)$ = dipole moment

$\dot{p}(t_0) = q\ddot{r}' = qa$ where, $a$ = acceleration of the ~~point~~ charge distribution.

$$\therefore P = \frac{\mu_0}{6\pi c} q^2 a^2 = \frac{\mu_0 q^2 a^2}{6\pi c} = \text{Larmor formula} \ldots\ldots (53) \quad [\text{Using } (52)].$$

$\Rightarrow P \propto a^2 \Rightarrow$ the power radiated by a point charge is proportional to the square of its acceleration.

Lets' know about physical interpretation of above calculations. What we have done above, a multipole expansion of the retarded potentials, carried to the lowest order in $r'$ ~~that is capable of~~ (first order in $r'$) that is capable of producing em radiation (field that go like $1/r$). This is the electric dipole term. Because total charge $Q$ is conserved, an electric monopole does not radiate (as $\vec{E} \sim 1/r^2$). If charge was not conserved, the first term in equation (37) would be,

$$\phi_{mono} = \frac{1}{4\pi\epsilon_0} \frac{Q(t_0)}{r} \quad \text{where, } \dot{Q}(t_0) \neq 0 \text{ because}$$

$Q(t_0)$ is time dependent.

Then, $$\vec{E}_{mono} = \vec{\nabla}\phi_{mono} = \frac{1}{4\pi\epsilon_0} \vec{\nabla}\frac{Q(t_0)}{r}$$

Now, $\vec{\nabla}\frac{1}{r} \sim 1/r^2 \Rightarrow$ Neglected.

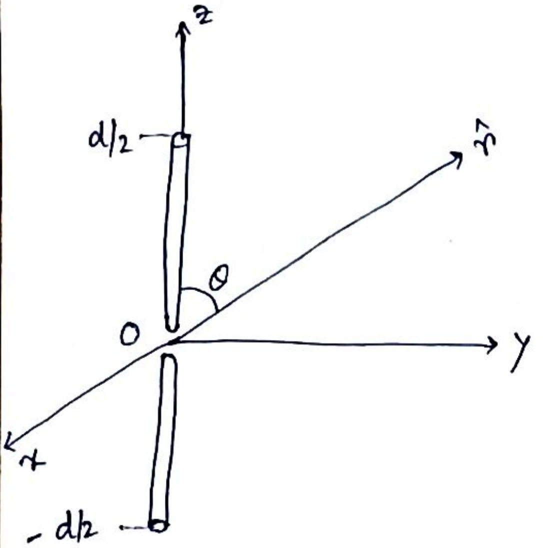
$$\vec{\nabla} Q(t_0) = \dot{Q}(t_0)\, \vec{\nabla} t_0 = -\frac{\dot{Q}(t_0)}{c}\hat{r} \quad \left[\because \vec{\nabla} t_0 = \vec{\nabla}(t - r/c) = -\frac{1}{c}\hat{r}\right].$$

$$\therefore \vec{E}_{mono} = -\frac{\dot{Q}(t_0)\hat{r}}{4\pi\epsilon_0 c r}$$

If somehow, the electric dipole moment gets vanished, then there is no electric dipole radiation and one must look to the next term: the second order terms of $r'$ ($r'^2$ terms). This term can be separated into two parts, one of which is related to the 'magnetic dipole moment' of the source & the other ~~term~~ part is 'electric quadrupole moment' of the source. If the magnetic dipole & electric quadrupole contributions vanish anyway, then we have to consider $r'^3$ terms & this can be separated into two parts also, one yields 'magnetic quadrupole' and the other corresponds to 'electric octapole' radiation and so on.

## Linear, Center-fed antenna :- [Ref: Jackson, Panofsky & Phillips]

Linear antenna is a straight wire carrying a time-varying current which emits radiation. A centerfed antenna has a little gap at the centre and current is fed at the center. A simple example of an electric dipole radiator is a center-fed, linear antenna whose length d is small compared to a wavelength. The antenna is assumed to be oriented along the z-axis, extending from $z = (d/2)$ to $z = -(d/2)$ with a narrow gap at the center (at origin) (Fig.1). The current is in the same direction in each half of the antenna, having a value $I'$ at the gap and falling approximately linearly to zero at the ends. Hence the current density can be written as,



Short, linear center-fed antenna

$$\vec{J}(\vec{r}') = I' \sin\left[\left(\frac{kd}{2}\right) - k|z|\right] \delta(x)\,\delta(y)\,\hat{z} \quad \cdots (i)$$

for $|z| < d/2$

The delta functions assure that the current flows only ~~at~~ along the z-axis. $I'$ is the peak value of the current if $kd \geq \pi$.

The current at the gap $= I_g = I' \sin\left(\frac{kd}{2}\right) \quad \cdots (ii)$

Let us discuss about different conditions.

1) Short antenna or long wave-length limit :- $kd \ll 1 \Rightarrow d/\lambda \ll 1 \Rightarrow d \ll \lambda$

$$\vec{J}(\vec{r}') \simeq I'\left[\frac{kd}{2} - k|z|\right] = I'\cdot\frac{kd}{2}\left[1 - \frac{2|z|}{d}\right] = I_0\left(1 - \frac{2|z|}{d}\right) \quad \cdots (iii)$$

$$I_0 = I'\cdot\frac{kd}{2} \quad \cdots (iv)$$

2) $kd = \pi \Rightarrow$ Half-wave antenna.

$$\Rightarrow kd = \pi \Rightarrow \frac{2\pi}{\lambda} d = \pi \Rightarrow d = \lambda/2 \quad \cdots (v)$$

3) Full-wave antenna :- $kd = 2\pi \Rightarrow \frac{2\pi}{\lambda} d = 2\pi \Rightarrow d = \lambda . \quad \cdots (vi)$

We know that $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\,\frac{e^{ikR}}{R}\,dv' \quad \cdots (vii)$

where, $R = |\vec{r} - \vec{r}'| = (r^2 + r'^2 - 2\vec{r}\cdot\vec{r}')^{1/2} \simeq r\left(1 - \frac{2\vec{r}\cdot\vec{r}'}{r^2}\right)^{1/2}$ [Neglecting $r'^2$]

$$\simeq r\left(1 - \frac{\vec{r}\cdot\vec{r}'}{r^2}\right) = r - \hat{r}\cdot\vec{r}' \quad \cdots (viii)$$

$$(vii) \Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\cdot\frac{e^{ikr}}{r}\int \vec{J}(\vec{r}')\,e^{-ik(\hat{r}\cdot\vec{r}')}\,dv' \quad \left[\frac{1}{R} \simeq \frac{1}{r}\right].$$

$$= \frac{\mu_0}{4\pi}\,\frac{e^{ikr}}{r}\,I'\hat{z}\int_{-d/2}^{+d/2} \sin\left[\left(\frac{kd}{2}\right) - k|z|\right] e^{ikz\cos\theta}\,dz.$$

$$= \frac{\mu_0}{4\pi}\,\frac{e^{ikr}}{r}\,I'\hat{z}\,\frac{2\left[\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)\right]}{k\sin^2\theta} \quad \cdots (ix)$$

11.

Equation of continuity : $\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$ ..... (x)

$\rho = \rho(\vec{r}')\, e^{-i\omega t} \Rightarrow \frac{\partial \rho}{\partial t} = (-i\omega)\rho$.

$\therefore$ (x) $\Rightarrow \frac{dJ(z)}{dz} + (-i\omega)\rho(z) = 0$

$\Rightarrow \frac{dJ(z)}{dz} = i\omega \rho(z)$ ... (xi)

For short-antenna, $\vec{J}(\vec{r}') = I_0\left(1 - \frac{2|z|}{d}\right)$.

$\frac{dJ(z)}{dz} = -\frac{2I_0}{d}$ for $z>0$ ..... (xii a)

$= +\frac{2I_0}{d}$ for $z<0$ ..... (xii b)

(xi) $\Rightarrow$ $\rho(z) = \frac{1}{i\omega}\left(-\frac{2I_0}{d}\right) = \frac{2iI_0}{\omega d}$ for $z>0$ ... (xiii a)

$= -\frac{2iI_0}{\omega d}$ for $z<0$ ..... (xiii b)

$\therefore$ The dipole moment is parallel to the z-axis and has the magnitude $p(z) = \int_{-d/2}^{+d/2} z\,\rho(z)\,dz = \pm \frac{2iI_0}{\omega d} \times 2 \int_0^{d/2} z\,dz$

$= \pm \frac{4iI_0}{\omega d} \times \frac{z^2}{2}\Big|_0^{d/2} = \pm \frac{4iI_0}{\omega d}\,\frac{d^2}{8} = \pm \frac{iI_0 d}{2\omega}$

$\therefore$ $p(z) = +\frac{iI_0 d}{2\omega}$ for $z>0$ .... (xiv a)

$= -\frac{iI_0 d}{2\omega}$ for $z<0$ .... (xiv b)

$p(\vec{r}', t_0) = p(\vec{r}')\, e^{-i\omega t_0} \Rightarrow \dot{p}(t_0) = (-i\omega)p(t_0) \Rightarrow \ddot{p}(t_0) = -\omega^2 p(t_0) \Rightarrow \ddot{p}^2 = \omega^4 p^2$ .... (xv)

Total radiated power $P(t) = \frac{\mu_0 \ddot{p}^2}{6\pi c}$ [Using (52)]

$= \frac{\mu_0}{6\pi c} \times \omega^4 p^2 = \frac{\mu_0}{6\pi c}\omega^4 \cdot \frac{I_0^2 d^2}{4\omega^2}$ [Using (xiv) & (xv)]

$= \frac{\mu_0 \omega^2 I_0^2 d^2}{24\pi c} = \frac{\mu_0 k^2 d^2 I_0^2 c}{24\pi} = \frac{\pi \mu_0 c I_0^2 d^2}{6\lambda^2}$ $[\because \omega = ck = 2\pi c/\lambda]$.

..... (xvi)

Time average of radiated power = $\langle P \rangle = \frac{1}{2} P(t)$

$= \frac{\mu_0 \omega^2 I_0^2 d^2}{48\pi c} = \frac{\mu_0 c I_0^2 k^2 d^2}{48\pi} = \frac{\pi \mu_0 c I_0^2 d^2}{12\lambda^2}$ ..... (xvii)

(xvi) $\Rightarrow$ $P(t) \propto \omega^2$

$\therefore$ For a fixed input current, the radiated power increases as the square of the frequency for short-antenna or long wave-length limit, $kd \ll 1$.

$$\langle P \rangle = \frac{1}{2} I_0^2 R_{rad} \quad \cdots\cdots (xviii)$$

where, $R_{rad}$ is known as radiation resistance of the antenna & it has dimension of a resistance. It is the total resistance of the antenna if the conductivity is perfect.

Comparing (xvii) & (xviii) we get,

$$R_{rad} = \frac{\mu_0 \omega^2 d^2}{24\pi c} = \frac{\mu_0 c K^2 d^2}{24\pi} = \frac{\pi \mu_0 c d^2}{6\lambda^2} \simeq 197 \,(d/\lambda)^2 \,\Omega \text{ for short antenna.} \quad \cdots\cdots (xix)$$

Directivity of an antenna (G) is defined as the ratio of maximum angular power delivered to the average power per unit solid angle.

$$G = \frac{\text{max. angular power delivered}}{\text{average power per unit solid angle}} = \frac{\left\langle \frac{dP}{d\Omega} \right\rangle_{max}}{\langle P \rangle / 4\pi} \quad \cdots\cdots (xx)$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = iK \,\hat{n} \times \vec{A} \Rightarrow |\vec{B}| = K \sin\theta \,|\vec{A}| \quad \cdots\cdots (xxi)$$

Angular power delivered = power radiated per unit solid angle $= \frac{dP}{d\Omega}$

Time average of $\frac{dP}{d\Omega} = \left\langle \frac{dP}{d\Omega} \right\rangle = \langle \vec{N} \rangle r^2$

Now, $N = \frac{c}{\mu_0} |\vec{B}_{rad}|^2$ [Using 48]

$$\langle \vec{N} \rangle = \frac{1}{2} N = \frac{c}{2\mu_0} |\vec{B}_{rad}|^2 = \frac{c}{2\mu_0} K^2 \sin^2\theta \,|\vec{A}|^2 \qquad \text{[Using (xxi)]}.$$

$$\therefore \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{cK^2}{2\mu_0} \sin^2\theta \left(\frac{\mu_0}{4\pi}\right)^2 \frac{I'^2}{r^2} \times \frac{4r^2}{K^2} \frac{\left|\cos\left(\frac{Kd}{2}\cos\theta\right) - \cos\left(\frac{Kd}{2}\right)\right|^2}{\sin^4\theta} \qquad \text{[Using (ix)]}$$

$$= \frac{c\mu_0 I'^2}{8\pi^2} \left| \frac{\cos\left(\frac{Kd}{2}\cos\theta\right) - \cos\left(\frac{Kd}{2}\right)}{\sin\theta} \right|^2 \quad \cdots\cdots (xxii) \qquad \left[\because \left\langle \frac{dP}{d\Omega} \right\rangle = \langle \vec{N} \rangle r^2\right]$$

Let us discuss three cases.

1) Short antenna limit $\Rightarrow Kd << 1$

$$\cos\left(\frac{Kd}{2}\cos\theta\right) \simeq 1 - \frac{\left(\frac{Kd}{2}\cos\theta\right)^2}{2!} = 1 - \frac{(Kd)^2}{8}\cos^2\theta \qquad \left[\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$$\cos\left(\frac{Kd}{2}\right) \simeq 1 - \frac{1}{2!}\left(\frac{Kd}{2}\right)^2 = 1 - \frac{(Kd)^2}{8}$$

$$\therefore \cos\left(\frac{Kd}{2}\cos\theta\right) - \cos\left(\frac{Kd}{2}\right) = \frac{(Kd)^2}{8}(1 - \cos^2\theta) = \frac{(Kd)^2}{8}\sin^2\theta.$$

$$\therefore \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c\mu_0 I'^2}{8\pi^2} \frac{(Kd)^4}{64} \sin^2\theta \quad \cdots\cdots (xxiii)$$

$$\therefore \langle P \rangle = \iint_{\theta\,\phi} \left\langle \frac{dP}{d\Omega} \right\rangle \sin\theta \,d\theta \,d\phi = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{c\mu_0 I'^2}{8\pi^2} \frac{(Kd)^4}{64} \cdot \sin^3\theta \,d\theta \,d\phi$$

$$\cancel{\left[\int_0^{\pi} \sin^3\theta \,d\theta = \frac{4}{3} \;\&\; \int_0^{2\pi} d\phi = 2\pi\right]}$$

$$= \frac{c\mu_0 I'^2}{8\pi^2} \times \frac{4}{3} \times 2\pi \times \frac{(Kd)^4}{64} \qquad \left[\because \int_{\theta=0}^{\pi} \sin^3\theta \,d\theta = 4/3 \;\&\; \int_{\phi=0}^{2\pi} d\phi = 2\pi\right]$$

$$= \frac{c\mu_0 I'^2 (Kd)^4}{64 \times 3\pi} = \frac{c\mu_0 I_0^2}{48\pi} (Kd)^2 \quad \cdots (xxiv) \quad \text{[Using (iv)]}$$

13.

We can see that, (xvii) & (xxiv) have same values.

$$\left\langle \frac{dP}{d\Omega} \right\rangle_{max} = \frac{c\mu_0 I_0^2}{8\pi^2} \times \frac{(kd)^2}{16} \quad \left[ \text{Using (xxiii) \& max. value of } \sin^2\theta = 1 \right]$$

$$\therefore G = \frac{\left\langle \frac{dP}{d\Omega} \right\rangle_{max}}{\langle P \rangle / 4\pi} = \frac{\frac{c\mu_0 I_0^2}{8\pi^2} \times \frac{(kd)^2}{16}}{\frac{c\mu_0 I_0^2}{48\pi} \times \frac{(kd)^2}{4\pi}} = \frac{\frac{1}{8\times 16\pi^2}}{\frac{1}{48\times 4\pi^2}} = \frac{48\times 4\pi^2}{8\times 16\pi^2} = \frac{3}{2} = 1.5$$

2) Half-wave antenna :- $kd = \pi$ ~~$\Rightarrow d$~~

$$\therefore \left\langle \frac{dP}{d\Omega} \right\rangle_{hw} = \frac{c\mu_0 I'^2}{8\pi^2} \frac{\left|\cos\left(\frac{\pi}{2}\cos\theta\right)\right|^2}{\sin^2\theta} \quad \left[ \text{Using (xxii) \& } kd = \pi \right]$$

$$\left\langle \frac{dP}{d\Omega} \right\rangle_{hw} \Big|_{max} = \frac{c\mu_0 I'^2}{8\pi^2} \quad \left[ \text{for } \theta = \pi/2 \right].$$

$$\langle P \rangle_{hw} = \frac{c\mu_0 I'^2}{8\pi^2} \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \underbrace{\frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \sin\theta \, d\theta}_{=1.2188} \, d\phi = \frac{c\mu_0 I'^2}{8\pi^2} \times 2\pi \times 1.2188 = \frac{1}{2} I'^2 R_{rad}\big|_{hw}$$

$$\therefore R_{rad}\big|_{hw} = \frac{c\mu_0}{2\pi} \times 1.2188 = \frac{\mu_0}{\sqrt{\mu_0 \epsilon_0}} \frac{1.22}{2\pi} = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1.22}{2\pi} \simeq 73\,\Omega.$$

$$G_{hw} = \frac{\langle dP/d\Omega \rangle_{hw}\big|_{max}}{\langle P \rangle_{hw} / 4\pi} = \frac{\frac{c\mu_0 I'^2}{8\pi^2}}{\frac{c\mu_0 I'^2}{4\pi} \times 1.22 \times \frac{1}{4\pi}} = \frac{1}{8\pi^2} \times \frac{16\pi^2}{1.22} \simeq 1.64.$$

3) Full-wave antenna :- $kd = 2\pi$.

$$\therefore \left\langle \frac{dP}{d\Omega} \right\rangle_{fw} = \frac{c\mu_0 I'^2}{8\pi^2} \frac{\left|\cos(\pi\cos\theta) + 1\right|^2}{\sin^2\theta} = \frac{c\mu_0 I'^2}{8\pi^2} \frac{4\cos^4\left[(\pi/2)\cos\theta\right]}{\sin^2\theta}.$$

$$\left\langle \frac{dP}{d\Omega} \right\rangle_{fw} \Big|_{max} = \frac{c\mu_0 I'^2}{8\pi^2} \times 4 \quad \left[ \text{for } \theta = \pi/2 \right].$$

$$\langle P \rangle_{fw} = \frac{c\mu_0 I'^2}{8\pi^2} \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \underbrace{\frac{4\cos^4\left[(\pi/2)\cos\theta\right]}{\sin^2\theta} \sin\theta \, d\theta}_{=0.8295} \, d\phi = \frac{c\mu_0 I'^2}{8\pi^2} \times 2\pi \times 4 \times 0.8295 = \frac{1}{2} I'^2 R_{rad}\big|_{fw}$$

$$R_{rad}\big|_{fw} = \frac{2c\mu_0}{\pi} \times 0.8295 = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{2\times 0.8295}{\pi} \simeq 199\,\Omega.$$

$$G_{fw} = \frac{\frac{c\mu_0 I'^2}{8\pi^2} \times 4}{\frac{c\mu_0 I'^2}{\pi} \times 0.8295 \times \frac{1}{4\pi}} = \frac{4}{8\pi^2} \times \frac{4\pi^2}{0.8295} \simeq 2.41$$

$$\left\langle \frac{dP}{d\Omega} \right\rangle_{fw} \Big|_{max} = 4 \times \left\langle \frac{dP}{d\Omega} \right\rangle_{hw} \Big|_{max} \quad \text{at } \theta = \pi/2.$$

$\therefore$ Full-wave antenna delivers 4 times power than that of the half-wave antenna. For $\theta = \pi/2$, full-wave antenna can be expressed as the superposition of two half-wave antennas.