VIVEKANANDA COLLEGE

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NAAC ACCREDITED ‘A’ GRADE

Topic: Classical Field Theory

Course Title: Classical Electrodynamics

Paper: PHY 421 (M.Sc)

Unit: 3.A

Semester: II

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Name of the Department: Physics

Classical Field Theory (KG)

In this note we will generalize the familiar principle of least action in classical mechanics of point particles to continuous systems and fields. Both the later class contain an uncountable number of degrees of freedom (d.o.f) compared to the case of particle mechanics which can at most contain a countable infinite number of particles and hence a countable infinite number of d.o.f. It is expected that we will have the following generalization of the Lagrangian:

$$\Sigma L_i = \int L(x)dx \tag{1}$$

To begin, we consider longitudinal wave propagation through an elastic bar of **unit cross sectional area** lying along the X -axis. We will exclude complications associated with the boundary conditions by considering the bar to be of infinite length. In the usual derivation of the elastic wave equation in the bar, we first break the bar into a large number of little blocks of length Δx and focus on what is happening to any one of them as shown in **fig.1** (attached at the end). The little block $ABCD$ is small but contains a large number of molecules so that statistical fluctuations are negligible.

The displacement of the vertical sections from their equilibrium positions (position at rest when there is no wave) are given by $\eta(x)$. Here η denotes the generalized coordinates, while the coordinate x is an index that distinguish the vertical sections at different positions along the X -axis. Look at the fig.1. Thus, the generalized coordinates are the displacements of different vertical sections from their equilibrium positions of no wave. Compare this with particle mechanics where the generalized coordinates are given by q_i, where i takes different integral values to distinguish different generalized coordinates q. Thus, in continuous mechanics x is no longer a generalized coordinate but plays the role of an index that distinguish between different generalized coordinates. Since x can take a continuous range of values, we call the corresponding mechanics as continuous mechanics or mechanics of continuum.

The block shown in the picture is large enough to apply the laws of elasticity like the Hook's law. The kinetic energy of the block is given by:

$$\Delta T = \frac{1}{2}(\mu\Delta x)(\frac{\partial\eta(x)}{\partial t})^2 \tag{2}$$

where μ is the mass per unit length in the equilibrium condition and $\mu\Delta x$ is the mass of the little element $ABCD$. We have considered here only the first order term.

The little block undergoes a strain due to the difference in $\eta(x)$ and $\eta(x+\Delta x)$. This stores some potential energy within the block. The potential energy is given by the elements of the rod to the right of the face BC. Each block is coupled to the neighboring block. Within the elastic limit, the potential energy per unit length is given by: $\frac{1}{2}(stress)(strain) = \frac{1}{2}Y(strain)^2$, where Y is the Young's modulus. We have applied the Hook's law to obtain this expression. Here, the strain in the little block is given by:

$$\frac{\eta(x+\Delta x)-\eta(x)}{equilibrium\ length} = \frac{\eta(x+\Delta x)-\eta(x)}{\Delta x} = \frac{\Delta\eta(x)}{\Delta x} \tag{3}$$

The potential energy stored in the little block is given by:

$$\Delta V = \frac{1}{2}(Y\Delta x)(\frac{\Delta\eta}{\Delta x})^2 \tag{4}$$

We can define the total kinetic energy, total potential energy and the Lagrangian for the complete rod as the sum of the corresponding quantities for every little block:

$$\begin{aligned} T &= \sum\frac{1}{2}(\mu\Delta x)(\frac{\partial\eta}{\partial t})^2 \\ V &= \sum\frac{1}{2}(Y\Delta x)(\frac{\Delta\eta}{\Delta x})^2 \\ L &= \sum[\frac{1}{2}\mu(\frac{\partial\eta}{\partial t})^2 - \frac{1}{2}Y(\frac{\Delta\eta}{\Delta x})^2]\Delta x \end{aligned} \tag{5}$$

It is not trivial to replace the summation in the above expressions by integrals over x, in particular for the practical case when the length of the bar is finite. The situation is similar to that of defining mass density or charge density of an extended body. We assume that there exists little blocks whose size is very small compared to the size of the bar yet large enough to include a large number of molecules so that we can apply the Hook's law to these little blocks. Similar assumptions are used in the Lagrangian description of every field theory. We now replace Δx in the above expressions by dx where dx is small compared to the dimension of the complete bar but is not a mathematically infinitesimal quantity that can be made arbitrarily small. We replace the summation sign in the expression for the Lagrangian by integral signs to obtain:

$$\begin{aligned} L &= \int [\frac{1}{2}\mu(\frac{\partial\eta}{\partial t})^2 - \frac{1}{2}Y(\frac{\partial\eta}{\partial x})^2]dx = \int \mathscr{L} dx \\ \mathscr{L}[\frac{\partial\eta}{\partial t},\frac{\partial\eta}{\partial x}] &= \frac{1}{2}\mu(\frac{\partial\eta}{\partial t})^2 - \frac{1}{2}Y(\frac{\partial\eta}{\partial x})^2 \end{aligned} \tag{6}$$

$\mathscr{L}$ is known as the *Lagrangian Density.* We define the action as:

$$S = \int L dt = \int\int \mathscr{L} dx dt \tag{7}$$

We now proceed to derive the Euler-Lagrange's equation from the above action. In particle mechanics, Euler-Lagrange's equation gives us the trajectory along which the time integral of the Lagrangian is an extremum which is usually the minimum. To find this path we compare the value of the action integral for two trajectories that are infinitesimally close to one another and coincide at the terminal points. We show the situation in **fig.2**. Thus, we consider two paths as:

$$q(t), q(t) + \delta q(t);\ \delta q(t) = 0 \ \ for\ t = t_1,\ t = t_2 \tag{8}$$

Where t_1, t_2 are the starting and finishing instatns of the motion. $q(t)$ is taken to be the actual path for which the action is minimum while $q(t) + \delta q(t)$ gives the varied path. $\delta q(t)$ is the variation from the actual path at time t. It is same as virtual displacement. In case of the action given by Eq.(7), we assume that the actual vibration of the bar is given by a function $\eta(t, x)$ for which the action is minimum. However, $\eta(t, x)$ is now dependent on two parameters t and x that take continuous values. Thus, we find that the position coordinate x now plays the role of a parameter similar to time t. We now find the corresponding Euler-Lagrange's equation, the solution of which gives us the actual configuration of the vibrating bar. We consider a more general problem by generalizing the Lagrangian density in Eqs.(6,7) to:

$$\begin{aligned} S &= \int\int \mathscr{L} dx dt \\ \mathscr{L} &\equiv \mathscr{L}[\eta, \dot{\eta}, \eta'],\ \ \dot{\eta} = \frac{\partial\eta}{\partial t},\ \eta' = \frac{\partial\eta}{\partial x} \end{aligned} \tag{9}$$

Notice that $\mathscr{L}$ in the above equation depends on η which was not the case with Eq.(6). As in the case of particle mechanics, let $\eta(t, x)$ give the configuration of the system for which the action in the above equation is minimum. By configuration we mean that $\eta(t, x)$ as a function of x and t gives us the displacements of the vertical sections of the bar for all values of time starting from an initial time to a final time. We now consider another configuration $\eta(t, x) + \delta\eta(t, x)$, which differs from $\eta(t, x)$. The Lagrangian density for the varied configuration is related to that of the actual configuration by the following simple relation:

$$\begin{aligned} \mathscr{L} + \delta\mathscr{L} &= \mathscr{L} + \frac{\partial\mathscr{L}}{\partial\eta}\delta\eta + \frac{\partial\mathscr{L}}{\partial\dot{\eta}}\delta\dot{\eta} + \frac{\partial\mathscr{L}}{\partial\eta'}\delta\eta' \\ &= \mathscr{L} + \frac{\partial\mathscr{L}}{\partial\eta}\delta\eta + \frac{\partial\mathscr{L}}{\partial\dot{\eta}}\frac{\partial(\delta\eta)}{\partial t} + \frac{\partial\mathscr{L}}{\partial\eta'}\frac{\partial(\delta\eta)}{\partial x} \end{aligned} \tag{10}$$

where in the first step we have neglected the higher order terms. In the second step, we have interchanged the order of taking partial derivatives and taking variations. This is similar to the particle mechanics case, the only difference is that we are using it for two parameters t and x. The change in the action as we move from the actual configuration to the varied configuration is given by:

$$\begin{aligned}
\delta S &= \int dxdt[\frac{\partial \mathscr{L}}{\partial \eta}\delta\eta \;+\; \frac{\partial \mathscr{L}}{\partial \dot\eta}\frac{\partial(\delta\eta)}{\partial t} + \frac{\partial \mathscr{L}}{\partial \eta'}\frac{\partial(\delta\eta)}{\partial x}] \qquad (11)\\
&= \int dxdt[\frac{\partial \mathscr{L}}{\partial \eta}\delta\eta \;-\; \frac{\partial}{\partial t}(\frac{\partial \mathscr{L}}{\partial \dot\eta})\delta\eta - \frac{\partial}{\partial x}(\frac{\partial \mathscr{L}}{\partial \eta'})\delta\eta + \frac{\partial}{\partial t}(\frac{\partial \mathscr{L}}{\partial \dot\eta}\delta\eta) + \frac{\partial}{\partial x}(\frac{\partial \mathscr{L}}{\partial \eta'}\delta\eta)]\\
&= \int dxdt[\frac{\partial \mathscr{L}}{\partial \eta}\delta\eta \;-\; \frac{\partial}{\partial t}(\frac{\partial \mathscr{L}}{\partial \dot\eta})\delta\eta - \frac{\partial}{\partial x}(\frac{\partial \mathscr{L}}{\partial \eta'})\delta\eta]\\
&+ \int dx[(\frac{\partial \mathscr{L}}{\partial \dot\eta}\delta\eta)(t_2,x) \;-\; (\frac{\partial \mathscr{L}}{\partial \dot\eta}\delta\eta)(t_1,x)] + \int dt[(\frac{\partial \mathscr{L}}{\partial \eta'}\delta\eta)(t,x_2) - (\frac{\partial \mathscr{L}}{\partial \eta'}\delta\eta)(t,x_1)]\\
&= \int dxdt[\frac{\partial \mathscr{L}}{\partial \eta} \;-\; \frac{\partial}{\partial t}(\frac{\partial \mathscr{L}}{\partial \dot\eta}) - \frac{\partial}{\partial x}(\frac{\partial \mathscr{L}}{\partial \eta'})]\delta\eta
\end{aligned}$$

Where, in the second step we have used identities like: $\frac{\partial}{\partial t}(\frac{\partial \mathscr{L}}{\partial \dot\eta}\delta\eta) = \frac{\partial}{\partial t}(\frac{\partial \mathscr{L}}{\partial \dot\eta})\delta\eta + \frac{\partial \mathscr{L}}{\partial \dot\eta}\frac{\partial(\delta\eta)}{\partial t}$. In the third step we have omitted the boundary terms by using the conditions that $\delta\eta(t_2,x) = \delta\eta(t_1,x) = \delta\eta(x_2,t) = \delta\eta(x_1,t) = 0$. Apart from the initial conditions, these conditions also indicate that the variations are vanishing at the spatial boundaries x_2 and x_1. In practical cases, the system is ultimately confined within some finite region and we can assume that the fields are always vanishing at the boundary of this region. Thus, both the actual configuration and the varied configuration are vanishing at the boundary and we have: $\delta\eta(x_2,t) = \delta\eta(x_1,t) = 0$. It is now easy to derive the Euler-Lagrange's equation. If we assume that the actual configuration is the configuration for which the first order variation in the action is vanishing for all small variations in $\eta(t,x)$, then we have from Eq.(11):

$$\frac{\partial \mathscr{L}}{\partial \eta} - \frac{\partial}{\partial t}(\frac{\partial \mathscr{L}}{\partial \dot\eta}) - \frac{\partial}{\partial x}(\frac{\partial \mathscr{L}}{\partial \eta'}) = 0 \qquad (12)$$

This step is similar to the corresponding step for particle mechanics. The above equation gives the Euler-Lagrange's equation for a continuous system like a vibrating bar with only longitudinal oscillation. For longitudinal wave propagation through a uniform bar, the Lagrangian density is given by Eq.(6). We have:

$$\frac{\partial \mathscr{L}}{\partial \eta} = 0, \quad \frac{\partial \mathscr{L}}{\partial \dot\eta} = \mu\frac{\partial \eta}{\partial t}, \quad \frac{\partial \mathscr{L}}{\partial \eta'} = Y\frac{\partial \eta}{\partial x}. \qquad (13)$$

The Euler-Lagrange's equation then gives us the following wave equation:

$$\frac{\partial^2 \eta}{\partial x^2} = (\frac{\mu}{Y})\frac{\partial^2 \eta}{\partial t^2} \qquad (14)$$

This is the familiar equation for longitudinal wave propagation through an elastic bar. The velocity of the wave is $\sqrt{Y/\mu}$. We can generalize the formalism to higher dimension as well as consider fields that can have more than one component. To illustrate, we consider the following examples:

1. **Schrodinger Equation:** (details attached)
2. **Relativistic Scalar Field:** (details attached)

This is an example where the number of parameters are four that includes time and three spatial dimensions (t,x,y,z). The Lagrangian density of a relativistic scalar field is given by:

$$\begin{aligned}
\mathscr{L} &= \frac{1}{2}c^2(\partial_\mu\phi\partial^\mu\phi) - \frac{1}{2}m^2c^2\phi^2 \qquad (15)\\
&= \frac{1}{2}\dot\phi^2 - \frac{1}{2}c^2\vec\nabla\phi.\vec\nabla\phi - \frac{1}{2}m^2c^2\phi^2
\end{aligned}$$

Here we have used the metric signature $(+,-,-,-)$. The Euler-Lagrange's equation give us the Klien-Gordon equation for the scalar field:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + m^2\phi = 0 \qquad (16)$$

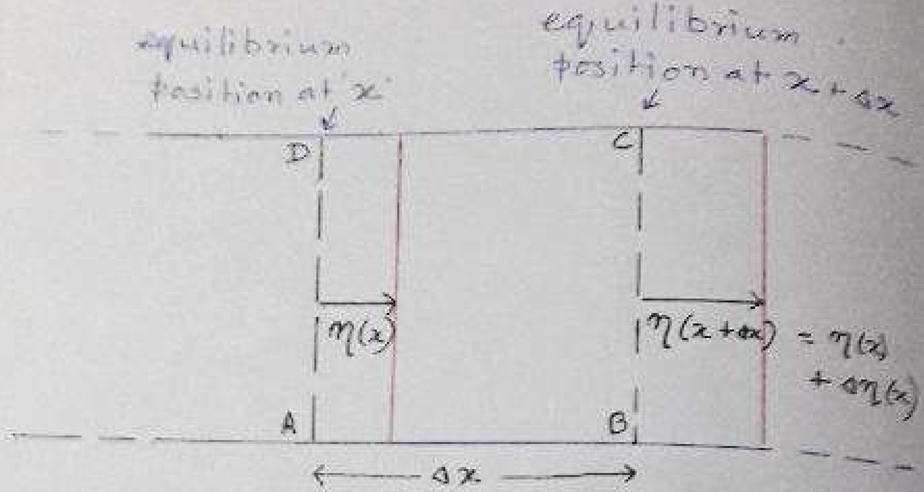
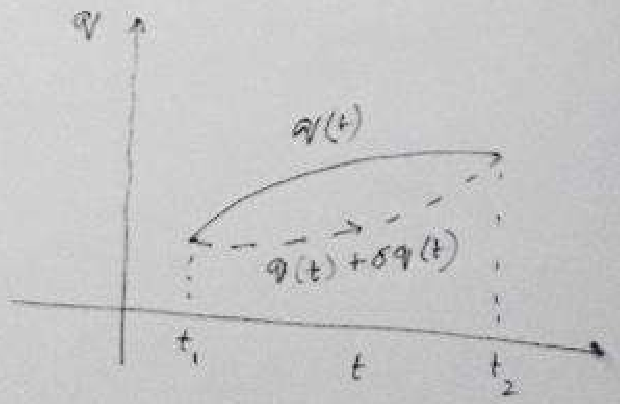

equilibrium position at 'x'
equilibrium position at x+Δx
D
C
A
B
η(x)
η(x+Δx) = η(x) + Δη(x)
Δx

fig. 1


q
q(t)
q(t) + δq(t)
t₁
t
t₂

Ex. 1 Schrodinger equation:

We can derive the Schrodinger equation from the principle of least action using a suitable action, which we choose to be:

$$S = \iint dx\,dt \left[\frac{1}{2} i\hbar \left(\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right) + \frac{\hbar^2}{2m} \left(\frac{\partial \psi}{\partial x} \right) \left(\frac{\partial \psi^*}{\partial x} \right) \right] \quad -①$$

We are considering the one-dim. case only. Here we have two independent fields: ψ and ψ^* (the cont. conj. of ψ). We will obtain two Euler-Lagrange equations corresponding to ψ and ψ^*. We consider first the EL eq. for ψ^*. We have,

$$\mathcal{L} = \frac{i\hbar}{2} \left(\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right) + \frac{\hbar^2}{2m} \left(\frac{\partial \psi}{\partial x} \right) \left(\frac{\partial \psi^*}{\partial x} \right) \quad -②.$$

$$\frac{\partial \mathcal{L}}{\partial \psi^*} = -\frac{i\hbar}{2} \frac{\partial \psi}{\partial t}, \quad \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}^*} \right) = \frac{i\hbar}{2} \frac{\partial \psi}{\partial t}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi^*}{\partial x}\right) = \frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}$$

Combining with the EL eqn. for ψ^*:

$$\frac{\partial \mathcal{L}}{\partial \psi^*} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}^*}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \psi^{*\prime}}\right) = 0$$

where, $\dot{\psi}^* = \frac{\partial \psi^*}{\partial t}$,

$\psi^{*\prime} = \frac{\partial \psi^*}{\partial x}$.

we have,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}.$$

The equation for 'ψ' gives the complex conjugate of this equation as expected.

Ex. 2 Relativistic Scalar Field:

This is the simplest case of a relativistic field theory. We consider three spatial dimensions. Being relativistic, we use the four dimensional Minkowski space to describe the dynamics. The action is taken to be a scalar ~~in the~~ similar to the case of a relativistic point particle. We take it to be:

$$S = \int d^4x \left[\frac{1}{2} c^2 (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 c^2 \phi^2 \right] \quad -①$$

where, $d^4x = (dx^0)(dx^1)(dx^2)(dx^3)$

$$= c\,dt\,dx\,dy\,dz$$

the four-dimensional invariant volume element. The 1st term is:

$$\partial_\mu \phi\, \partial^\mu \phi = \partial_0 \phi\, \partial^0 \phi + \partial_1 \phi\, \partial^1 \phi + \cdots$$

$$\cancel{= \left(\frac{1}{c}\frac{\partial \phi}{\partial t}\right)\left(\frac{1}{c}\frac{\partial \phi}{\partial t}\right)}$$

$$= \frac{\partial\phi}{\partial x^0}\frac{\partial\phi}{\partial x_0} + \frac{\partial\phi}{\partial x^1}\frac{\partial\phi}{\partial x_1} + \cdots$$

$$= \left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right)\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right) - \left(\frac{\partial\phi}{\partial x}\right)^2 - \cdots$$

We have used a metric with signature $(+,-,-,-)$,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and, $\cancel{x_0}$ $x_\alpha = \eta_{\alpha\beta} x^\beta$

$$\Rightarrow x_0 = x^0, \quad x_i = -x^i,$$

$$\frac{\partial\phi}{\partial x^1} = \frac{\partial\phi}{\partial x}, \text{ etc.} \quad x^i = (x, y, z)$$

Thus the Lagrangian density is,

$$\mathcal{L} = \frac{c^2}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2c^2\phi^2$$

$$= \frac{1}{2}c^2\left[(\partial_0\phi)^2 - (\partial_x\phi)^2 - (\partial_y\phi)^2 - (\partial_z\phi)^2\right] - \frac{1}{2}m^2c^2\phi^2. \quad -②$$

We now derive the corresponding Euler-Lagrange's equation. In this case there are more than one spatial coordinates

and the EL equation is given by:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \cancel{\frac{\partial \mathcal{L}}{}} \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial_x \phi)}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial(\partial_y \phi)}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \mathcal{L}}{\partial(\partial_z \phi)}\right) = 0$$

$$\Rightarrow m^2 c^2 \phi - \frac{\partial \phi}{\partial t^2} + \cancel{\phi}\, c^2\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right] = 0. \quad -(3)$$

We have used, :

$$\partial_0 \phi = \frac{1}{c}\frac{\partial \phi}{\partial t}.$$

Thus, we have,

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + m^2 c^2 \phi = 0$$

We write it as:

$$\Box \phi + m^2 \phi = 0$$

$$\Rightarrow (\Box + m^2)\phi = 0. \quad -(4)$$

$\Box \equiv \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi$, is known as the ~~D'Alemberetian~~ d'Alembertian operator.

Eq. (4) is known as the Klien-Gordon equation.

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where, $d^4x = (dx^0)(dx^1)(dx^2)(dx^3)$

$$= c\,dt\,dx\,dy\,dz$$

the four-dimensional invariant volume element. The 1st term is:

$$\partial_\mu \phi\, \partial^\mu \phi = \partial_0 \phi\, \partial^0 \phi + \partial_1 \phi\, \partial^1 \phi + \cdots$$

$$= \cancel{\left(\frac{1}{c}\frac{\partial \phi}{\partial t}\right)\left(\frac{1}{c}\frac{\partial \phi}{\partial t}\right)}$$

$= \frac{\partial\phi}{\partial x^0}\frac{\partial\phi}{\partial x_0} + \frac{\partial\phi}{\partial x^1}\frac{\partial\phi}{\partial x_1} + \cdots$

$= \left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right)\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right) - \left(\frac{\partial\phi}{\partial x}\right)^2 - \cdots$

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$$\Rightarrow m^2 c^2 \phi - \frac{\partial \phi}{\partial t^2} + \cancel{\phi} \; c^2\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right] = 0 . \quad - (3)$$

We have used, i.e.

$$\partial_0 \phi = \frac{1}{c}\frac{\partial \phi}{\partial t} .$$

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