VIVEKANANDA COLLEGE THAKURPUKUR KOLKATA-700063

NAAC ACCREDITED ‘A’ GRADE

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Classical Field Theory:2 (KG)

In this note, we will describe the Hamiltonian formalism of non-relativistic field theory. We will follow an old version of the classical mechanics book of Goldstein. If any one does not have access to this book, you can tell me so that I can give you a few pages personally. Recall from the first note that we can describe the dynamics of a continuous system like a vibrating bar when sound wave is propagating through it or an extended system like a relativistic scalar field by the principle of least action through a suitable Lagrangian density and action. In this note, we first generalize the notation to describe the motion in three dimensions and fields with more than one components. We will consider the non-relativistic case only. We will the notation: x^α to denote (t, x, y, z), x^i to denote (x, y, z). Note that $x^0 = t$ (not ct), and $x^i = x_i$ in the non-relativistic case. We had considered an oscillating bar with longitudinal oscillation in the previous note. The field η has only one component since the oscillation is longitudinal. The Lagrangian density was given by:

$$\mathscr{L}[\partial_t\eta, \partial_x\eta] = \frac{1}{2}\mu(\frac{\partial\eta}{\partial t})^2 - \frac{1}{2}Y(\frac{\partial\eta}{\partial x})^2 \tag{1}$$

Note that the bar is not coupled to any external system and hence the Lagrangian density is independent of x^μ. In general the system under consideration can be coupled to some fixed external sources like an electromagnetic field coupled to a fixed source. The Lagrangian density can also be dependent on the fields themselves. Including all generalizations, we have the following action to describe a general field:

$$\begin{aligned} S &= \int dtdxdydz\mathscr{L}[\eta_\rho(x^\alpha), \eta_{\rho,\alpha}(x^\alpha), x^\alpha], \quad \eta_{\rho,\alpha} = \frac{\partial\eta}{\partial x^\alpha} = \partial_\alpha\eta_\rho \\ &= \int d^4x\mathscr{L}[\eta_\rho(x^\alpha), \eta_{\rho,\alpha}(x^\alpha), x^\alpha] = \int Ldt \\ L &= \int dxdydz\mathscr{L}[\eta_\rho(x^\alpha), \eta_{\rho,\alpha}(x^\alpha), x^\alpha] = \int d^3x\mathscr{L}[\eta_\rho(x^\alpha), \eta_{\rho,\alpha}(x^\alpha), x^\alpha] \end{aligned} \tag{2}$$

Several cooments are in order:

Firstly, we will always use a single integral sign to express multiple integrals. How many integrations are there should be read from the number of infinitesimals present under the integral sign. For example, there are four integrals for d^4x and three **spatial** integrals for d^3x

Secondly, we will denote the space and time index by the Greek letters that appear at the beginning of the Greek alphabet (x^α), while we will use the Greek letters that appear at the end of the Greek alphabet to denote the field components like η_ρ. Purely spatial indices will be denoted by the Latin indices: x^i, (x, y, z).

Thirdly, the Lagrangian density depends on the functions η_ρ and its various derivatives. We will use the third bracket to denote such dependence.

Last, we will assume that $\mathscr{L}$ is only quadratically dependent on the various derivatives of η_ρ.

Eq.(2) leads to the following Euler-Lagrange's equations:

$$\frac{\partial\mathscr{L}}{\partial\eta_\rho} - \frac{\partial}{\partial x^\alpha}(\frac{\partial\mathscr{L}}{\partial\eta_{\rho,\alpha}}) = 0 \tag{3}$$

We now proceed to construct a Hamiltonian description of the fields. To do so, we consider the Hamiltonian for a system of N particles:

$$H = \sum p_k\dot{q}_k - L, \quad p_k = \frac{\partial L}{\partial\dot{q}_k} \tag{4}$$

In our case the Lagrangian is given by:

$$L = \int d^3x\mathscr{L}[\eta_\rho(x^\alpha), \eta_{\rho,\alpha}(x^\alpha), x^\alpha] \tag{5}$$

We can consider this Lagrangian as the sum of the Lagrangians due to an infinite number of generalized coordinates η_ρ indexed by ρ and (x,y,z). For every value of ρ, there are an infinite number of generalized coordinates labeled by different possible values of (x,y,z). As an example, we consider the Maxwell-Proca Lagrangian:

$$\begin{aligned} L &= \int d^3x \mathscr{L} = \int d^3x(-\frac{1}{4}F^2 + \frac{1}{2}m^2A^2), \\ A^2 &= A_\mu A^\mu, \quad A_\mu \equiv (A_0, A_1, A_2, A_3), \\ F^2 &= F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \tag{6}$$

Note that we have a sum over all the field components A_μ and a sum over all possible values of position with the later is expressed by the integral over d^3x. Returning to the general case given by Eq.(2), we define a canonically conjugate momentum density for every generalized coordinates using $\mathscr{L}$ as:

$$\pi^\rho(x^\alpha) = \frac{\partial \mathscr{L}(t,x^i)}{\partial \eta_{\rho,t}(t,x^i)} \tag{7}$$

This is called a **density** because it is a derivative of the Lagrangian **density**. It is important to note that by using the time derivative, we have singled out time from the complete set of space-time coordinates: (t,x,y,z). This is not much significant in non-relativistic field theories. In these cases time is absolute and we can distinguish time from the spatial coordinates in any inertial frame. However, this can be inconvenient in relativistic field theories like the Maxwell-Proca theory that describe the relativistic kinematics of massive photons. In this course we will only consider the Hamiltonian formalism of the non-relativistic fields. Also note that there is no difference between π^ρ and π_ρ in non-relativistic field theory.

We now define the Hamiltonian density of the field as:

$$\mathscr{H} = \pi^\rho \dot{\eta}_\rho - \mathscr{L}, \quad \dot{\eta}_\rho = \frac{\partial \eta_\rho}{\partial t} \tag{8}$$

Note that $\dot{\eta}_\rho$ are no longer independent variables. They are dependent on $(\eta_\rho, \pi_\rho, \eta_{\rho,i}, x^\alpha)$ through the defining equation (7). This is similar as in particle mechanics. The Hamiltonian is defined as a spatial integral over the spatial region in which the field is confined:

$$H = \int d^3x \mathscr{H} \tag{9}$$

We now continue to derive the canonical equations of motions. As in the case of particle mechanics, we express the time derivative of the Hamiltonian density in two alternate ways:

$$\frac{d\mathscr{H}}{dt} = \dot{\pi}^\rho \dot{\eta}_\rho + \pi^\rho \ddot{\eta}_\rho - \frac{\partial \mathscr{L}}{\partial \eta_\rho}\dot{\eta}_\rho - \frac{\partial \mathscr{L}}{\partial \dot{\eta}_\rho}\ddot{\eta}_\rho - \frac{\partial \mathscr{L}}{\partial \eta_{\rho,i}}\dot{\eta}_{\rho,i} - \frac{\partial \mathscr{L}}{\partial t} \tag{10}$$

The second and the fourth terms of the above equation cancels each other due to the definition given by Eq.(7). We thus have,

$$\frac{d\mathscr{H}}{dt} = \dot{\pi}^\rho \dot{\eta}_\rho - \frac{\partial \mathscr{L}}{\partial \eta_\rho}\dot{\eta}_\rho - \frac{\partial \mathscr{L}}{\partial \eta_{\rho,i}}\dot{\eta}_{\rho,i} - \frac{\partial \mathscr{L}}{\partial t} \tag{11}$$

However, as mentioned below Eq.(8), $\mathscr{H}$ can also be regarded as a function of $(\eta_\rho, \pi_\rho, \eta_{\rho,i}, x^\alpha)$. We can express $\frac{d\mathscr{H}}{dt}$ as the time derivatives of these quantities directly. We then have,

$$\frac{d\mathscr{H}}{dt} = \frac{\partial \mathscr{H}}{\partial \pi^\rho}\dot{\pi}^\rho + \frac{\partial \mathscr{H}}{\partial \eta_\rho}\dot{\eta}_\rho + \frac{\partial \mathscr{H}}{\partial \eta_{\rho,i}}\dot{\eta}_{\rho,i} + \frac{\partial \mathscr{H}}{\partial t} \tag{12}$$

Comparing Eqs.(11,12), we derive the following relations:

$$\begin{aligned}
\dot{\eta}_\rho &= \frac{\partial \mathscr{H}}{\partial \pi^\rho} \\
\frac{\partial \mathscr{L}}{\partial \eta_\rho} &= -\frac{\partial \mathscr{H}}{\partial \eta_\rho} \\
\frac{\partial \mathscr{L}}{\partial \eta_{\rho,i}} &= -\frac{\partial \mathscr{H}}{\partial \eta_{\rho,i}} \\
\frac{\partial \mathscr{L}}{\partial t} &= -\frac{\partial \mathscr{H}}{\partial t}
\end{aligned} \tag{13}$$

The first equation gives us one of the canonical pair of equations. To obtain the other equation, we proceed as follows:

$$\begin{aligned}
\frac{\partial \mathscr{H}}{\partial \eta_\rho} &= -\frac{\partial \mathscr{L}}{\partial \eta_\rho} \\
&= -\frac{\partial}{\partial x^\alpha}(\frac{\partial \mathscr{L}}{\partial \eta_{\rho,\alpha}}) \\
&= -\dot{\pi}^\rho - \frac{\partial}{\partial x^i}(\frac{\partial \mathscr{L}}{\partial \eta_{\rho,i}}) \\
&= -\dot{\pi}^\rho + \frac{\partial}{\partial x^i}(\frac{\partial \mathscr{H}}{\partial \eta_{\rho,i}})
\end{aligned} \tag{14}$$

Where, in the first step we have used Eq.(13 b), in the second step we have used the Euler-Lagrange's equation given by Eq.(3) and in the last step we have used Eq.(13 c). Thus, we have the following canonical equations of motion:

$$\begin{aligned}
\dot{\eta}_\rho &= \frac{\partial \mathscr{H}}{\partial \pi^\rho} \\
\dot{\pi}^\rho &= -\frac{\partial \mathscr{H}}{\partial \eta_\rho} + \frac{\partial}{\partial x^i}(\frac{\partial \mathscr{H}}{\partial \eta_{\rho,i}})
\end{aligned} \tag{15}$$

We can now express the time rate change of the Hamiltonian density using the canonical equations in the following way:

$$\begin{aligned}
\frac{d\mathscr{H}}{dt} &= \frac{\partial \mathscr{H}}{\partial \pi^\rho}\dot{\pi}^\rho + \frac{\partial \mathscr{H}}{\partial \eta_\rho}\dot{\eta}_\rho + \frac{\partial \mathscr{H}}{\partial \eta_{\rho,i}}\dot{\eta}_{\rho,i} + \frac{\partial \mathscr{H}}{\partial t} \\
&= \frac{\partial}{\partial x^i}(\dot{\eta}_\rho \frac{\partial \mathscr{H}}{\partial \eta_{\rho,i}}) + \frac{\partial \mathscr{H}}{\partial t}
\end{aligned} \tag{16}$$

This is as close as we can get to the corresponding particle mechanics part $\frac{dH}{dt} = \frac{\partial H}{\partial t}$. We can define fundamental Poisson bracket between the canonical variables (η_ρ, π^ρ) in the following way:

$$\{\eta_\rho(t, x^i), \pi^\sigma(t, y^i)\} = \delta^\sigma_\rho \delta(x^i - y^i) \tag{17}$$

Where δ^σ_ρ denotes the Kronecker delta and $\delta(x^i - y^i)$ denotes the Dirac delta function.