VIVEKANANDA COLLEGE THAKURPUKUR KOLKATA-700063

NAAC ACCREDITED ‘A’ GRADE

Topic: Classical Field Theory

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Notations:

In the following we will represent matrices by the capital letters either as $[A]$ or A. Only the matrix representing the metric tensor will be given by g. Transposes will be given by $\tilde{A}$. Four vectors will be given by the boldfaces like $\mathbf{x}$. The corresponding column matrices will be given by just x. Summations over the space-time indices will be given by the Greek letters while the summations over the spatial indices will be given by the Roman letters. Inner product of the four vectors will be given by either $\mathbf{x}.\mathbf{y}$ or just as $x.y$. Spatial vectors will be given by $\vec{x}$.

I. A MATHEMATICAL FORMULATION OF THE MINKOWSKI SPACE

In the low-velocity Newtonian mechanics space and time are separately assumed to be absolute. This had been an outcome of the experiments associated with the motions of material particles when the velocities are small compared to that of the light in vacuum. The corresponding relativity principles which relate the space time coordinates of a given event as is observed in two different inertial frames are known as the Galilean principle of relativity.

If we synchronize two set of clocks at rest in two different inertial frames such that both the set assign the same value of time for a given event, the value of time assigned by each set to any other event will be the same. Thus we will have $t = t'$ valid for any event and time can be regarded as absolute.

The spatial coordinates of a given event as is observed in two different inertial frames are related by the well known relation: $x' = x - vt$, where v is the velocity of the frame F' w.r.t F in the positive X-direction of F. In general x' is different from x and we find space can not be absolute in the same way as time is. Still the concept of absolute space was introduced by Newton to explain the *inertial forces*. Any observer, in an accelerated motion w.r.t the absolute space, experiences the inertial forces. An example is an accelerating train. It had been a long standing issue to locate an inertial frame which is at rest w.r.t the absolute space and is thereby distinct compared to the other inertial frames.

We have learnt from the Special theory of Relativity that we have to discard the concept of absolute space and absolute time. The first postulate of Einstein ruled out the existence of any prejudiced inertial frame. The second postulate predicted that **simulteneity** is a relative concept and time no longer remain absolute.

Herman Minkowski, a Polish mathematician, combined the two postulates of Einstein into a single mathematical axiom. The axiom is that "*all natural laws must be expressible as tensor field equations on a (flat) absolute space-time manifold*". A tensor field equation is a mathematical formulation of the Physical laws which remains to be of the same form in all the inertial frames. To illustrate, one can consider the inhomogeneous equation satisfied by the electromagnetic field tensor. In a particular inertial frame this equation is given as: $\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu$. In another inertial frame this equation becomes: $\partial_\nu F'^{\mu\nu} = -\frac{4\pi}{c} j'^\mu$. Although the magnitudes of the physical quantities can be different in the two frames, the form of the equations remain the same. Thus all the reference frames are equivalent as far as the relations between the different physical quantities measured in a single frame are concerned. The absoluteness here refers to the fact that the space-time remains unaffected by the presence of matter and its motions.

This absolute space-time is referred to as the Minkowski space. Physical quantities are represented by scalars, vectors and tensorial quantities in this space. The three dimensional vector space of the position vectors in the Newtonian mechanics are generalized to the four dimensional vector space known as the Minkowski vector space. The elements of this vector space are the four dimensional position vectors whose components are the space-time coordinates of a particular event. We now give an axiomatic formulation of the Minkowski space.

The Minkowski vector space V is a vector space defined over the field of the real numbers R. The elements of V satisfy the following axioms.

Axioms of addition:

A1. $\mathbf{a} + \mathbf{b} \in V \quad \forall \mathbf{a},\mathbf{b} \in V$

A2. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \quad \forall \mathbf{a},\mathbf{b} \in V$

A3. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) \quad \forall \mathbf{a},\mathbf{b},\mathbf{c} \in V$

A4. There is $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{a} = \mathbf{a} \quad \forall \mathbf{a} \in V$

A5. For all $\mathbf{a} \in V$ there is $\mathbf{-a} \in V$ such that $(\mathbf{-a}) + \mathbf{a} = \mathbf{0}$

Axioms of multiplication by scalars:

M1. $\alpha\mathbf{a} \in V \quad \forall \mathbf{a} \in V, \ \forall \alpha \in R$

M2. $\alpha(\beta\mathbf{a}) = \alpha\beta\mathbf{a} \quad \forall \mathbf{a} \in V, \ \forall \alpha, \beta \in R$

M3. $1\mathbf{a} = \mathbf{a} \quad \forall \mathbf{a} \in V$

M4. $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b} \quad \forall \mathbf{a},\mathbf{b} \in V, \ \forall \alpha \in R$

M5. $(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a} \quad \forall \mathbf{a} \in V, \ \forall \alpha, \beta \in R$

Axioms of product:

Apart from the axioms of addition and the axioms for multiplication by the scalars, there are a set of axioms of product. Thus the Minkowski space is a normed vector space similar to the ordinary vector space of the three dimensional position vectors although the structure of the inner product (or the scalar product) in the Minkowski space is different from that of the ordinary vector space of the three dimensional position vectors as we will find in the following.

P1. $\mathbf{a}.\mathbf{b} \in R \quad \forall \mathbf{a},\mathbf{b} \in V$

P2. $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a} \quad \forall \mathbf{a},\mathbf{b} \in V$

P3. $\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c} \quad \forall \mathbf{a},\mathbf{b},\mathbf{c} \in V$

P4. *The axiom of nondegeneracy*: $\mathbf{a}.\mathbf{x} = 0 \quad \forall \mathbf{x} \in V$ if and only if $\mathbf{a} = \mathbf{0}$.

This axiom makes the Minkowski space different from the ordinary vector space of the three dimensional position vectors. The norm of a three dimensional position vector is positive definite (> 0) and P4 is replaced by,

$\vec{r}.\vec{r} \geq 0$ and the equality holds if and only if $\vec{r} = \vec{0}$.

The norm of a three dimensional position vector is given by $\vec{r}.\vec{r} = x^2 + y^2 + z^2$. thus the norm is always positive and the only case when the norm can be zero is the case when all the three coordinates are vanishing which is the null vector. The ordinary vector space of the three dimensional position vectors is known as the Euclidean vector space and is denoted by E_3. Axiom P4 indicates that merely $\mathbf{a}.\mathbf{a} = 0$ does not assure us that $\mathbf{a} = \mathbf{0}$. The norm of a four vector in the Minkowski space with non-zero components may be zero. A four vector may be said to be the zero vector (all the components are zero) if it has vanishing inner product with all the other four vectors. Thus four vectors in the Minkowski space can have positive, zero or negative norms. A four vector which has zero norm is usually known as a null vector. The vector with all the components vanishing is known as the zero null vector or the zero vector. This axiom is the *first axiom* that distinguishes V from E_3.

In addition to the above axioms we impose the *axiom of dimensionality* on the Minkowski space:

D1. Dim[V] = 4

Let us now introduce a basis in the Minkowski space. Since the dimension of V is 4 we can introduce a set of four linearly independent four vectors $\{\hat{e_1}, \hat{e_2}, \hat{e_3}, \hat{e_4}\}$ in the Minkowski space such that any vector can be expressed as a linear combination of these bases:

$$\mathbf{a} = a^1\hat{e_1} + a^2\hat{e_2} + a^3\hat{e_3} + a^4\hat{e_4} \tag{1}$$

The norm of a four vector is given by,

$$\mathbf{a}.\mathbf{a} = a^\alpha a^\beta(\hat{e_\alpha}.\hat{e_\beta}) = a^\alpha a^\beta g_{\alpha\beta} \tag{2}$$

Where we have introduced the metric tensor whose components in the basis $\{\hat{e_1}, \hat{e_2}, \hat{e_3}, \hat{e_4}\}$ are given by the following expression:

$$g_{\alpha\beta} = \hat{e_\alpha}.\hat{e_\beta} \tag{3}$$

Let us consider the 4×4 matrix g whose components are $g_{\alpha\beta}$. From the axiom P2, we find that g is a symmetric matrix $g_{\alpha\beta} = g_{\beta\alpha}$. Hence the eigenvalues of g are real. We next consider the characteristic equation:

$$|g - \lambda I| = 0 \tag{4}$$

Where $|A|$ means the determinant of the matrix A and λ are the eigenvalues.

The above equation indicates that we can find a basis $\{\hat{\epsilon_\mu}\}$ in V_4 such that the metric g is represented by a diagonal matrix in terms of these bases,i.e, $\hat{\epsilon_\mu}.\hat{\epsilon_\nu} = \lambda_\mu\delta_{\mu\nu}$

From the axiom of nondegeneracy it follows that $\lambda_\mu \neq 0$.

[Note: To show this let us assume that one of the eigenvalue, $\lambda_1 = 0$. Then we have, $\hat{\epsilon_1}.\hat{\epsilon_1} = 0$. Hence for all $\mathbf{x} \in V_4$ we have,

$\mathbf{x}.\hat{\epsilon_1} = 0$ as $\mathbf{x}$ can be expressed as $\mathbf{x} = \sum x^\mu \hat{\epsilon_\mu}$. From the axiom of degeneracy we then have, $\hat{\epsilon_1} = \mathbf{0}$.]

The signs of λ_μ can be positive or negative. We introduce the *axiom of signature*:

S1. $\lambda_1 > 0; \lambda_2, \lambda_3, \lambda_4 < 0$

[Note: The choice $\lambda_1, \lambda_2 > 0; \lambda_3, \lambda_4 < 0$ which is consistent with P4 and D1 is not adopted.]

The *axiom of nondegeneracy*, the *axiom of dimensionality* and the *axiom of signature* are the new physical input to the mathematical structure of space-time.

We define two four vectors to be M-orthogonal if,

$$\mathbf{a}.\mathbf{b} = 0 \tag{5}$$

We now state the following theorem without proof:

Theorem1: There exist an M-orthonormal basis $\{\hat{e_1}, \hat{e_2}, \hat{e_3}, \hat{e_4}\}$ in V_4 such that,

$$g_{\alpha\beta} = \hat{e_\alpha}.\hat{e_\beta} = d_{\alpha\beta} \tag{6}$$

where

$$d_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{7}$$

is the Lorentz metric.

The separation of a four vector is defined as $\sigma(\vec{a}) = \sqrt{\vec{a}\vec{a}}$.

II. INADEQUACY OF THE CONCEPT OF LENGTH AND ANGLE IN THE MINKOWSKI SPACE

You are familiar with the timelike, null and spacelike vectors in the Minkowski space which indicates the inadequacy of the concept of length in the Minkowski space. We now discuss the inadequacy of the concept of angle in the Minkowski space.

Let us consider a set of vector in V_4: $\mathbf{u} = \hat{e_1}$, $\mathbf{v_n} = \hat{\mathbf{e_1}} + \frac{\mathbf{n-1}}{\mathbf{n}}\hat{\mathbf{e_4}}$; where n is a positive integer.

We then have the following expression as the angle between $(\mathbf{u},\mathbf{v}_n)$:

$$\begin{aligned} \cos(\mathbf{u}, \mathbf{v_n}) &= \frac{(\mathbf{u}.\mathbf{v_n})}{\sqrt{(\mathbf{u}.\mathbf{u})(\mathbf{v_n}.\mathbf{v_n})}} \\ &= \frac{n}{\sqrt{(2n-1)}} \end{aligned} \tag{8}$$

Which tends to infinity as n increases and hence the concept of angle between two vectors become meaningless. We can also consider the null vectors which are M-orthogonal to themselves. Thus the condition of M-orthogonality is just a name.

Three Theorems: The following three theorems have numerous applications in Quantum Field Theory and also in General Theory of Relativity. Here we only state them. Look at the class notes for a proof of the first.

Theorem1: No two timelike vectors in V_4 can be M-orthogonal.

Corollary: Let $\mathbf{u}$,$\mathbf{v}$ be two timelike four vectors in V_4 with both $u_0, v_0 > 0 (< 0)$. Then $\mathbf{u}.\mathbf{v} > 0$.

Theorem2: A timelike vector can not be M-orthogonal to a non-zero null vector.

Theorem3: Two non-zero null vectors are M-orthogonal if and only if they are scalar multiple of each other.

These theorems are important in Quantum Field Theory. We give two examples:

1. To discuss the Pauli-Lubanski vector, important to describe the intrinsic angular momentum (spin) of the elementary particles.
2. The decomposition of the four potential in the Scalar, Longtudinal and transverse components w.r.t the null wave vector.

You can find the details in the text book of Itzykson-Zuber.