VIVEKANANDA COLLEGE THAKURPUKUR KOLKATA-700063

NAAC ACCREDITED ‘A’ GRADE

Topic: Complex Analysis

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Complex Analysis

In this note we will discuss continuity and differentiability of complex functions that we had defined in the introductory class. We will also discuss harmonic functions.

1. Continuity of Complex Functions

We recall that we define a complex function to be a function of the complex variable: $z = x + iy$. We denote them by $f(z)$ or $w(z)$. Examples are:

$f(z) = z^2 = u(x, y) + iv(x, y) = x^2 - y^2 + i(2xy)$

$f(z) = z\bar{z} = u(x, y) + iv(x, y) = x^2 + y^2$

$f(z) = \exp(z) = u(x, y) + iv(x, y) = e^x(\cos y + i \sin y)$

As we have written in the above expressions, we can consider a complex function to be a transformation from the two dimensional complex $z(x, y)$-plane to the two dimensional complex $w(u, v)$ plane. For a given function $f(z)$, $u(x, y), v(x, y)$ become definite functions of (x, y). Look at **fig.1** attached with the note. We can learn a lot about the functions if we consider the level (equipotential) surfaces in the $z(x, y)$ -plane that give $u(x, y) = constant$ or $v(x, y) = constant$. To illustrate, if we consider the second function of the above examples, we find that the imaginary part is vanishing, while all points on a circle centered at the origin in the $z(x, y)$ -plane give us a constant value of $u(x, y)$. Look at **fig.2**.

Do you think $z, \bar{z}$ are independent?

Open Set: A subset U of the complex plane is open if whenever $z \in U$ there is a disk centered at z that is contained in U.

It is easy to view open sets in the complex plane. We draw any closed curve on the complex plane. The inner region that you obtain by deleting the boundary gives you an open set. A collection of open sets also give an open set.

Ex.1: The region given by $z\bar{z} < 4$. You can obtain this open set by considering the disk bounded by $x^2 + y^2 \leq 4$ and deleting the boundary $x^2 + y^2 = 4$. Note that the disc itself is not an open set. We consider a point z_0 on the boundary $x^2 + y^2 = 4$. If we now consider a small disk centered at z_0, it can not be contained in the original disc. Look at **fig.3**. The disk $x^2 + y^2 \leq 4$ is an example of a closed set.

Ex.2: The open half plane: $y > 0$. Note again the closed half plane, $y \geq 0$ is not open. Explain why.

Prob:1.1 Which among the following regions are open?
(a) $4 \leq z\bar{z} \leq 16$ (b) $4 < z\bar{z} \leq 16$ (c) $4 \leq z\bar{z} < 16$ (d) $4 < z\bar{z} < 16$

Continuity1: A complex function is continuous at a point z_0 if and only if,

$$lim_{z \to z_0} f(z) = f(z_0) = w_0 \tag{1}$$

Formally, for a given $\epsilon > 0$ we have a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

Note that in the real calculus of a single variable, we use the concept of right land limit and left hand limit to judge continuity. Thus, we say a real function $f(x)$ is continuous at $x = 0$ if $f(0^+) = f(0^-) = f(0)$. Here $f(0^+)$ is the value of $f(x)$ at $x = 0 + \epsilon, \epsilon > 0$ and $\epsilon \to 0$; while $f(0^-)$ is the value of $f(x)$ at $x = 0 - \epsilon, \epsilon > 0$ and $\epsilon \to 0$. To illustrate, $f(x) = x^2$ is continuous at $x = 0$.

In the case of complex functions, we can approach z_0 from any direction we want in the complex plane. For example, to study the continuity of $f(z) = z\bar{z}$ at $z = 0$, we can approach the origin in the z -plane along the positive real axis, negative real axis, positive imaginary axis, negative imaginary axis or any other direction that we want. The value of $f(z)$ should approach 0 irrespective of the direction of approach towards the origin in the complex z -plane. **This condition is essential to find continuity and differentiability of complex functions.**

Continuity2: A complex function $f(z)$ is said to be continuous in a region R if it is continuous at every point of R.

R can be an open set or a closed set. It may be neither open nor closed. In problem 1.1, (d) is an open set, (a) is a closed set while (b,c) are neither open nor closed.

Ex.3: Consider the function $w = z^2$. We study the continuity of w along the positive real axis. We express z in the polar form $z = re^{i\theta}$ and choose the usual range for θ, $0 \leq \theta < 2\pi$ with the positive real axis being $\theta = 0$. We then have, $w = r^2 e^{2i\theta}$. It is easy to find that w is continuous on the positive real axis since $f(x) = x^2$ is continuous and $e^{2i\epsilon} = e^{2i(2\pi-\alpha)} = 1$ when $\alpha \to 0$.

Ex.4: Consider the function $w = log(z) = log(r) + i\theta + i2n\pi$, where $n = 0, \pm1, \pm2, ...$ We get the principal branch when we choose $n = 0$. We consider the function $W = Log(z) = log(r) + i\theta$. It is easy to find that $Log(z)$ is not continuous at the positive real axis. The real part is continuous for any value of $r > 0$ but the imaginary part changes discontinuously across the positive real axis. It is zero on the real axis itself but it is close to 2π just below the positive real axis. The origin is known as the branch point of $log(z)$. From this example we learn **that a complex function is continuous at $z = z_0$ if both the real part and the imaginary part of the complex function are continuous at** $z = z_0$. Similar aspect is valid for a region in the complex plane.

Prob:1.2 Consider the function $w = +\sqrt{z}$. Show that it is not continuous at the positive real axis with the choice of θ given by Ex.4. Find a way to extend it to be continuous throughout the complex z -plane.

We conclude this section with a few theorems well known in ordinary calculus:

If we have two functions $f(z), g(z)$ each continuous at $z = z_0$ then,

1. $f + g$ is continuous at $z = z_0$
2. fg is continuous at $z = z_0$
3. $\frac{f}{g}$ is continuous at $z = z_0$ provided $g(z_0) \neq 0$.

2. Complex Differentiability of Complex Functions and Analytic Functions

The complex first derivative of a complex function is defined as:

$$f'(z_0) = lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{2}$$

provided the limit exists *i.e,* the limit is independent of the direction along which we approach z_0 in the complex z -plane. This is similar to the definition of continuity. We denote it in various alternate ways as:

$$f'(z_0) = \frac{df}{dz} = \frac{dw}{dz} = lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \tag{3}$$

We illustrate the above definition with two examples:

Ex.1: $f(z) = z^2$. In the derivative: $f'(z_0) = lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$, we can peak any choice of Δz, *e.g,* $\Delta z = \pm\Delta x, \pm i\Delta y$, in all the cases the result is $2z_0$. You can check it yourself easily. **Note that when we are approaching z_0 along the positive or negative imaginary axis, we are taking $\Delta z = \pm i\Delta y$ respectively. This is an important difference from calculus of two real variables, in which case you take $\Delta r = \pm\Delta y$ when you approach the point z_0 along the Y axis.** This is one of the principle difference between complex analysis and calculus of two real variables.

Ex.2: $f(z) = z\bar{z}$. We have:

$$\begin{aligned} \frac{df}{dz} &= \bar{z} + z\frac{\Delta \bar{z}}{\Delta z}, \quad \Delta z \to 0 \\ &= \bar{z} + z(\frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}) \\ &= \bar{z} + z, \quad for \quad \Delta z = \Delta x \\ &= \bar{z} - z, \quad for \quad \Delta z = i\Delta y \end{aligned} \tag{4}$$

In the second step, we have approached along the real axis. In the third step, we have approached along the imaginary axis. **Thus, we find that $f'(z)$ depends on the direction of approach and hence, $f'(z)$ does not exist any where apart from $z = 0$.**

Complex first derivative of complex functions satisfy a few properties similar to those in ordinary calculus of one real variable:

1. $(cf')(z) = cf'(z)$, where c is a complex constant.
2. $(f + g)'(z) = f'(z) + g'(z)$.
3. $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.
4. $(\frac{f}{g})' = \frac{f'(z)g(z)-f(z)g'(z)}{g^2(z)}$, provided $g(z) \neq 0$.

Analytic Function: A function $f(z)$ is analytic on an open set U if $f(z)$ is complex differentiable at every point of U.

When we say that a function $f(z)$ is analytic at a point $z = z_0$, we mean that $f(z)$ is analytic on an open set O containing z_0. Similarly, when we say that $f(z)$ is analytic on a region R, the function is analytic on an open set containing R. In the following we will always mean complex derivative by differentiation unless otherwise is mentioned.

Ex.3: We define a polynomial of order n in z as,

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + ... + a_1 z + a_0, \quad n \geq 0.$$

This is analytic on the the entire complex plane and is known as an **entire** function. Note that we exclude $|z| = \infty$ from the complex plane. When we include this we have the **extended complex plane**.

Ex.4: $f(z) = \exp{(z)}$, is another example of an entire function.

Ex.5: $f(z) = \frac{1}{z^2+1}$, is analytic on the entire complex plane apart from $z = \pm i$.

If a function $f(z)$ fails to be analytic at a point z_0 but is analytic on an open set containing z_0, then z_0 is known as an **isolated singular point** or **isolated singularity** of $f(z)$. In Ex.5, $z = \pm i$ are two isolated singular points of $f(z) = \frac{1}{z^2+1}$.

3. Cauchy-Riemann Equations and Harmonic Functions

Here, we want to discuss the conditions required to find whether a given function is analytic or not. These are given by the Cauchy-Riemann equations. We start with a definition:

Domain: An open set D is said to form a domain in the complex plane if we can join any pair of points within D by a curve staying within D.

To illustrate, the open disk $D : |z| < 2$ is a domain. The annular open set $A : 2 < |z| < 4$ is a domain. You can join any pair of points within each set by a curve staying within the corresponding set. On the other hand if we consider the union of these two sets, $C = D \bigcup A$, the resulting set is still an open set but it is not a domain. Any point in D can not be joined to any point in A by a curve staying completely in C since the circle $|z| = 2$ is not present in C. Look at **fig.4**.

We now state the necessary (and also sufficient) condtions for a complex function to be analytic on a domain.

Cauchy-Riemann Equations: Let $f(z) = u(x, y) + iv(x, y)$ be defined on a domain D in the complex plane, where u, v are real valued. Then $f(z)$ is analytic if and only if $u(x, y)$ and $v(x, y)$ have continuous first order partial derivatives with respect to x, y that satisfy the Cauchy-Riemann equations given by:

$$\begin{aligned} u_x(x,y) &= v_y(x,y) \\ u_y(x,y) &= -v_x(x,y) \end{aligned} \tag{5}$$

Where u_x means $\frac{\partial u}{\partial x}$, u_y means $\frac{\partial u}{\partial y}$ and the same is valid for v_x, v_y.

Proof: We only consider the first part of the theorem, which gives the above equations as the necessary conditions for a complex function to be analytic on a domain D. Let $f(z)$ be analytic on the domain D. Consider a point $z \in D$, we have:

$$f'(z) = lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{6}$$

We first consider Δz to be purely real, $\Delta z = \Delta x$. We then have from the above equation:

$$\begin{aligned} f'(z) &= lim_{\Delta x \to 0} \frac{f(z) + u_x \Delta x + i v_x \Delta x - f(z)}{\Delta x} \\ &= u_x + i v_x \end{aligned} \tag{7}$$

Where we have kept only the first order terms in first step since $\Delta x \to 0$. We next consider Δz to be purely imaginary, $\Delta z = i\Delta y$. Proceeding as above, we have from Eq.(6):

$$f'(z) = v_y - i u_y \tag{8}$$

If $f(z)$ is analytic, *i.e,* $f'(z)$ exists at a point z within D, then Eqs.(7,8) for $f'(z)$ should agree. We then obtain the Cauchy-Riemann equations by equating the real and imaginary parts of Eqs.(7,8).

It follows easily from Eq.(5) that u, v satisfy the Laplace's equation:

$$\begin{aligned} u_{xx}(x,y) + u_{yy}(x,y) &= 0 \\ v_{xx}(x,y) + v_{yy}(x,y) &= 0 \end{aligned} \tag{9}$$

Thus both $u(x, y)$ and $v(x, y)$ can be considered to represent electrostatic potentials in vacuum in two dimensions. Function which satisfy the Lapalace's equation in two dimension are known as **Harmonic** functions. We have not shown that the Cauchy-Riemann equations are sufficient for a complex function to be analytic on a domain. You can look at any text book for this part of the proof. This is not required now. Note that we can use the Taylor expansions in the above derivation provided we can join z and $z + \Delta z$ by a straight line staying within the region of interest. This is where we need a domain.

Prob:3.1 Show that e^z is analytic on the complex plane.

Complex functions.

$f : z \to w$

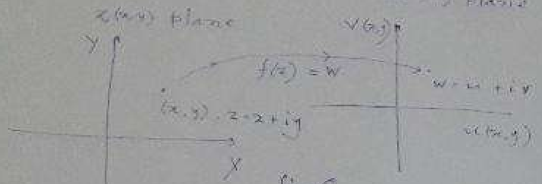
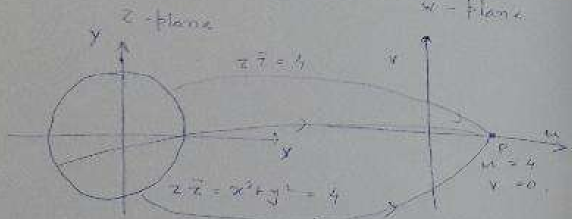
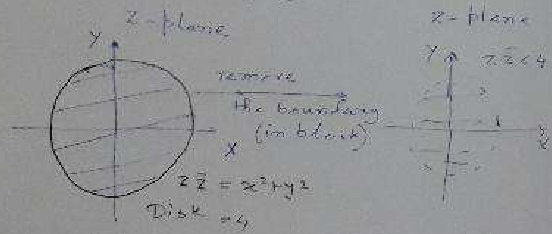


fig. ①

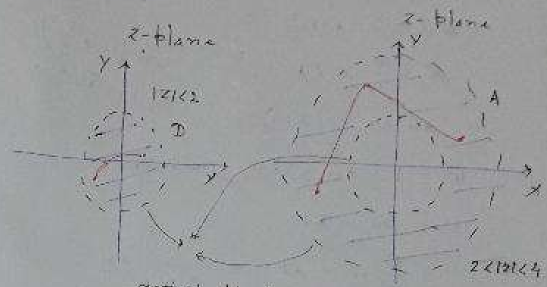


all points z on the circle to u = 4, v = 0.

fig. ②



Domain



dotted black curves are the boundaries removed to give open sets