VIVEKANANDA COLLEGE
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NAAC ACCREDITED ‘A’ GRADE

Topic: Born approximation and application

Course Title: SCATTERING THEORY

Paper: Quantum Mechanics II

Unit: PHY 422

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Name of the Department: PHYSICS

The Born Approximation

Let us first write the TISE

$$-\frac{\hbar^2}{2m}\cdot\nabla^2\psi + V\psi = E\psi$$

in operator form $\nabla^2\psi + k^2\psi = Q\psi$ — (1)

where $k^2 = \frac{2mE}{\hbar^2}$ and $Q = \frac{2m}{\hbar^2}\cdot V\psi$

This ~~the~~ is the spherical form of ~~Helmholtz~~ Helmholtz equation with a inhomogeneous term Q (depends on ψ) called the 'Source' term.

Now to solve the inhomogeneous equation we ~~have~~ better to go with Green's function ~~ten~~ technique.

General solution of the inhomogeneous equation consists of a sum of two components
(a) a general solution of homogeneous equation which is not other

$$(\nabla^2 + k^2)\psi_{hom}(\vec{r}) = 0$$

than the incident wave $\psi_{in} = e^{ikz}$ solution and another part is particular solution we will go To find particular solution technique through Green's function technique.

In this process ~~...~~ we have to find out a Green's function $G(\vec{r}-\vec{r}')$ such that the operator (∇^2+k^2) operate on the $G(\vec{r}-\vec{r}')$ and give a delta function $\delta(\vec{r}-\vec{r}')$

i.e ~~$(\nabla^2+k^2)G(\vec{r}) = \delta(\vec{r})$~~

$$(\nabla^2+k^2)\,G(\vec{r}-\vec{r}') = \delta^3(\vec{r}-\vec{r}') \quad — (2)$$

Then we can represent the solution

$$\psi(\vec{r}) = \int G(\vec{r}-\vec{r}_0)\, Q(\vec{r}_0)\, d^3(\vec{r}_0) \quad \text{—(3)}$$

This solution also satisfies SE

$$(\nabla^2+k^2)\psi(\vec{r}) = \int [(\nabla^2+k^2) G(\vec{r}-\vec{r}')] Q(\vec{r}') d\vec{r}'$$

$$= \int \delta^3(\vec{r}-\vec{r}')\, Q(\vec{r}')\, d\vec{r}'$$

$$(\nabla^2+k^2)\psi(\vec{r}) = Q(\vec{r}) \quad \text{(equation 1)}$$

To solve to find out the Green's function for Helmholtz equation. for simplicity we ~~we have to~~ have to go with fourier transform

say $(\vec{r}-\vec{r}') = \vec{r}$

Let $$G(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{s}\cdot\vec{r}}\, g(\vec{s})\, d^3\vec{s} \quad \text{— (4)}$$

Then $$(\nabla^2+k^2)\, G(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int (\nabla^2+k^2) e^{i\vec{s}\cdot\vec{r}}\, g(\vec{s})\, d^3s$$

$$\nabla^2 e^{i\vec{s}\cdot\vec{r}} = -s^2 e^{i\vec{s}\cdot\vec{r}}$$

$$(\nabla^2+k^2)\, G(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int (k^2-s^2)\, e^{i\vec{s}\cdot\vec{r}}\, g(\vec{s})\, d^3\vec{s}$$

and $$\delta^3(\vec{r}) = \frac{1}{(2\pi)^3} \int e^{i\vec{s}\cdot\vec{r}}\, d^3\vec{s} \quad \text{— (5)}$$

putting in $(\nabla^2+k^2)\, G(\vec{r}) = \delta^3(\vec{r})$

$$\frac{1}{(2\pi)^{3/2}} \int (k^2-s^2)\, e^{i\vec{s}\cdot\vec{r}}\, g(\vec{s})\, d^3\vec{s} = \frac{1}{(2\pi)^3} \int e^{i\vec{s}\cdot\vec{r}}\, d^3(\vec{s})$$

Using Plancherel's theorem we get

$$(k^2-s^2)\, g(s) \cdot \frac{1}{(2\pi)^{3/2}} = \frac{1}{(2\pi)^3}$$

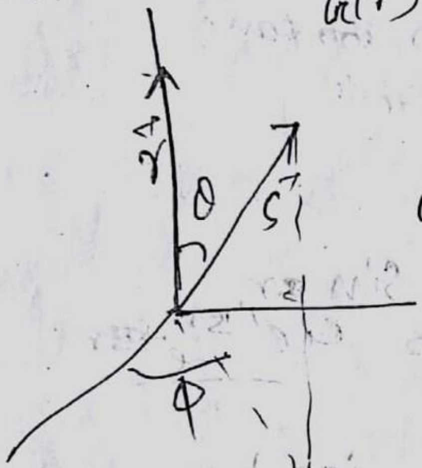
$$g(s) = \frac{1}{(2\pi)^{3/2}} \cdot \frac{1}{(k^2 - s^2)} \quad \text{— (6)}$$

puting the value of $g(s)$ in equation (4)

$$G(\vec{r}) = \frac{1}{(2\pi)^3} \int \frac{e^{i\vec{s}\cdot\vec{r}}}{(k^2 - s^2)} \cdot d^3 s \quad \text{— (7)}$$

To compute the integral (complex integral) we ~~too~~ should choose the ~~complex~~ spherical coordinate system -

$$G(\vec{r}) = \frac{1}{(2\pi)^3} \int_0^\infty s^2 ds \int_0^\pi \frac{e^{i\vec{s}\cdot\vec{r}}}{k^2 - s^2} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$



from the figure $\vec{r}\cdot\vec{s} = rs\cos\theta$

$$G(\vec{r}) = \frac{1}{(2\pi)^3} \int_0^\pi \frac{e^{isr\cos\theta} \sin\theta \, d\theta \int_0^{2\pi} d\phi}{(k^2 - s^2)}$$

$$= \frac{1}{(2\pi)^3} \, 2\pi \int_{+1}^{-1} \frac{e^{isr\cos\theta} (-d(\cos\theta))}{(k^2 - s^2)}$$

$$\therefore G(\vec{r}) = \frac{1}{(2\pi)^2} \int_{-1}^{1} \frac{e^{isr\cos\theta} \, d(\cos\theta)}{(k^2 - s^2)} \cdot \int_0^\infty s^2 ds$$

$$= \frac{1}{(2\pi)^2} \left. \frac{e^{isr\cos\theta}}{isr} \right|_{-1}^{1} \cdot \frac{1}{(k^2 - s^2)} \cdot \int_0^\infty s^2 ds$$

$$= \frac{1 \cdot \int_0^\infty s^2 ds}{(2\pi)^2 (k^2 - s^2)(sr)} \cdot 2\left(\frac{e^{isr} - e^{-isr}}{2i}\right)$$

$$= \int_0^\infty \frac{2\sin(sr) \cdot s^2 ds}{4\pi^2 (k^2 - s^2) \cdot (sr)}$$

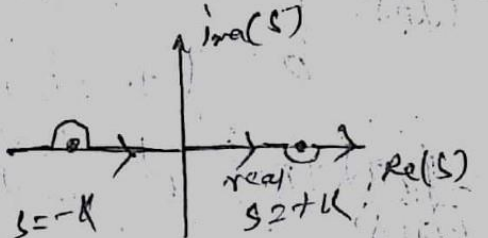
$$G(\vec{r}) = \frac{1}{(2\pi)^2} \cdot \frac{2}{r} \int_0^\infty \frac{s \sin(sr)}{(k^2 - s^2)} ds$$

To solve the integral we have to use the Cauchy's integral formula of residues

$$\oint \frac{f(z)}{(z-z_0)}\, dz = 2\pi i \cdot f(z_0)$$

where z_0 lies within the contour which has a pole.

But the poles are $\pm K$ on the real axis of the contour, so, we have to take one inside the contour & other outside, let us take $+K$ inside & $-K$ outside



$$\sin sr = \frac{e^{isr} - e^{-isr}}{2i}$$

$$G(r) = \frac{1}{(2\pi)^2} \cdot \frac{2}{r} \int_0^\infty \frac{s\left(e^{isr} - e^{-isr}\right)}{2i(K-s)(K+s)}\, ds$$

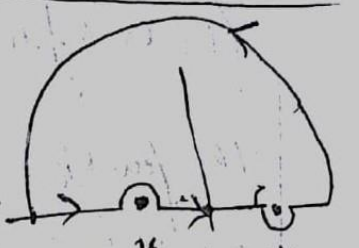
$$= \frac{1}{(2\pi)^2 \cdot 2ir}\left[\int_{-\infty}^{\infty} \frac{s e^{ikr}}{(K-s)(K+s)}\, ds - \int_{-\infty}^{\infty} \frac{s e^{-isr}}{(s+K)(K-s)}\, ds\right]$$

$$= \frac{1}{8\pi^2 r}(I_1 - I_2)$$

$$= \frac{i}{8\pi^2 r}\left[\int_{-\infty}^{\infty} \frac{s e^{isr}}{(s-K)(s+K)}\, ds - \int_{-\infty}^{\infty} \frac{s e^{-isr}}{(s+K)(s-K)}\, ds\right]$$

$$= \frac{i}{8\pi^2 r}\left(I_1 - I_2\right)$$

$$I_1 = \int_{-\infty}^{\infty} \frac{s e^{isr}}{(s-k)(s+k)} ds$$



$$= \oint \left[\frac{s e^{isr}}{(s+k)} \right] \frac{1}{s-k} ds$$

$$\equiv \oint \frac{f(z)}{(z-z_0)} ds \quad \text{where } f(z) = \frac{s e^{isr}}{(s+k)}$$

$$z_0 = k \quad z = s$$

Using Cauchy's integral formula

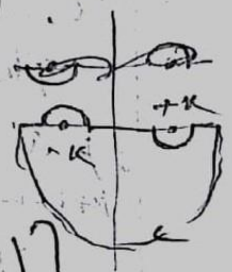
$$I_1 = 2\pi i \cdot f(z_0) = 2\pi i \cdot \frac{s e^{isr}}{s+k} \Bigg|_{s=k} = 2\pi i \cdot \frac{k e^{ikr}}{2k}$$

$$\boxed{I_1 = i\pi e^{ikr}}$$

Similarly $I_2 = \oint \left[\frac{s e^{-isr}}{(s-k)} \right] \frac{1}{s+k} ds$

$$= 2\pi i \left(\frac{s e^{-isr}}{s-k} \right) \Bigg|_{s=-k} = 2\pi i \cdot \frac{k e^{+ikr}}{(-2k)}$$

$$= -i\pi e^{ikr}$$



$$\therefore G(\vec{r}) = \frac{i}{8\pi^2 r} \left[i\pi e^{ikr} - (-i\pi e^{ikr}) \right]$$

$$= \frac{i(2i\pi e^{ikr})}{8\pi^2 r} = -\frac{e^{ikr}}{4\pi r} \quad \text{—8}$$

This is the Green's function for Helmholtz equation

Green's function $G_+(\vec{r}-\vec{r}\,') = -\frac{1}{4\pi} \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$

the function represents outgoing spherical wave from $\vec{r}\,'$ to the point where the effect is measured at $\vec{r}$.

So the equation $(\nabla^2+k^2)\psi(\vec{r}) = Q(\vec{r}) = \frac{2\mu}{\hbar^2} V(\vec{r})\psi(\vec{r})$ have solution

$$\psi(\vec{r}) = \psi_{inc}(\vec{r}) + \int G_+(\vec{r}-\vec{r}\,')\, Q(\vec{r}\,')\, d^3(\vec{r}\,')$$

$$= \psi_{inc}(\vec{r}) - \frac{2\mu}{\hbar^2 \cdot 4\pi} \int \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \cdot V(\vec{r}\,')\psi(\vec{r}\,')\, d^3_{\vec{r}\,'} \quad \text{— (9)}$$

This is an integral equation; it does not yet give the solution $\psi(\vec{r})$ but only contains it in the integrand.

All we have done is to rewrite the Schrodinger differential equation into an integral form. because integral form is more suitable to use in scattering theory.

The last equation can be solved by means of a series of successive or iterative approximation known as the Born series.

The zeroth order solution is given by $\psi_0(\vec{r}) = \psi_{inc}(\vec{r})$

The 1st order solution $\psi_1(\vec{r})$ is obtained by inserting $\psi_0(\vec{r}) = \psi_{in}(\vec{r})$ into the equation (9)

$$\psi_1(\vec{r}) = \psi_{inc}(\vec{r}) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}_1|}}{|\vec{r}-\vec{r}_1|} V(\vec{r}_1)\cdot\psi_0(\vec{r}_1)\, d^3r_1$$

$$= \psi_{inc}(\vec{r}) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}_1|}}{|\vec{r}-\vec{r}_1|} V(r_1)\psi_{inc}(\vec{r}_1)\, d^3r_1$$

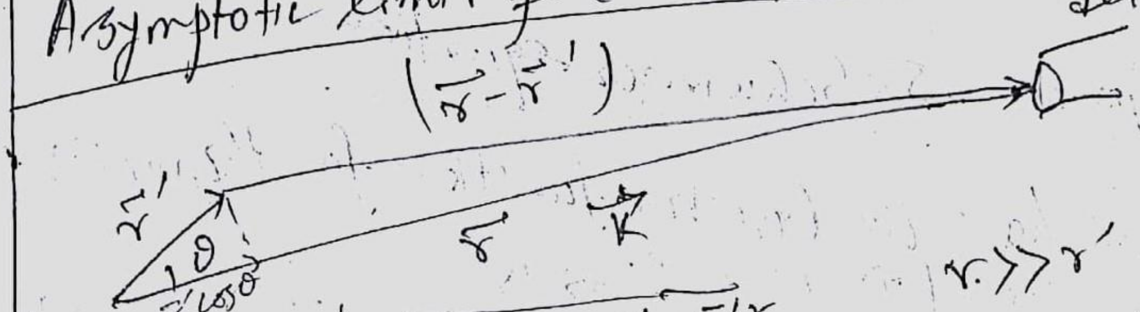
The 2nd order is obtained by replacing $\psi_0(\vec{r})$ by $\psi_1(\vec{r})$

$$\psi_2(\vec{r}) = \psi_{inc}(\vec{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}_2|}}{|\vec{r}-\vec{r}_2|} V(\vec{r}_2)\,\psi_1(\vec{r}_2)\,d^3r_2$$

$$= \psi_{inc}(\vec{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}_2|}}{|\vec{r}-\vec{r}_2|} V(\vec{r}_2)\,\psi_{inc}(\vec{r}_2)\,d^3r_2$$

$$+ \frac{m}{2\pi\hbar^2}\int \frac{e^{ik(\vec{r}-\vec{r}_2)}}{|\vec{r}-\vec{r}_2|} V(\vec{r}_2)\,d^3r_2 \int \frac{e^{ik|\vec{r}-\vec{r}_1|}}{|\vec{r}-\vec{r}_1|} V(\vec{r}_1)\,\psi_{inc}(\vec{r}_1)\,d^3r_1 \quad \text{--- (10)}$$

Asymptotic limit of the wave function



$r \gg r'$

$$k|\vec{r}-\vec{r}\,'| = k\sqrt{\vec{r}^2 - 2\vec{r}\cdot\vec{r}\,' + \vec{r}\,'^2}$$

$$= kr\left(\frac{r^2}{r^2} - \frac{2\vec{r}\,'}{r} + \frac{r'^2}{r^2}\right)^{1/2} \simeq kr\left(1 - \frac{2\vec{r}\,'}{r}\right)$$

$$= k\vec{r} - 2k\vec{r}\cdot\vec{r}\,' = kr - \vec{k}\cdot\vec{r}\,' \qquad \vec{k} = k\hat{r}$$

$\vec{k}$ = wave vector associated with the scattered wave

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\frac{1}{\left|1 - \frac{\vec{r}\cdot\vec{r}\,'}{r^2}\right|} \simeq \frac{1}{r}\left(1 + \frac{\vec{r}\cdot\vec{r}\,'}{r^2}\right) \simeq \frac{1}{r}$$

Putting this in equation — (9)

$$\psi(\vec{r}) = \psi_{inc} - \frac{m}{2\pi\hbar^2}\int \frac{e^{ikr}}{r}\,e^{-i\vec{k}\cdot\vec{r}\,'}\,V(\vec{r}\,')\,\psi(\vec{r}\,')\,d^3r'$$

Comparing with $\psi(\vec{r}) = A\left(\psi_{inc}(\vec{r}) + f(\theta,\phi)\,\frac{e^{ikr}}{r}\right)$

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2 A}\int e^{-i\vec{k}\cdot\vec{r}\,'}\,V(\vec{r}\,')\,\psi(\vec{r}\,')\,d^3r'$$

$$= -\frac{m}{2\pi\hbar^2 A}\langle \psi_{in}(\vec{r}\,')|V(\vec{r})|\psi(\vec{r}\,')\rangle$$

$$\psi_{inc}(\vec{r}\,') = e^{-i\vec{k}\cdot\vec{r}\,'} = e^{-ikr'\cos\theta}$$

So the differential cross section

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2 = \frac{m^2}{4\pi^2\hbar^4}\left|\int e^{-i\vec{k}.\vec{r}'} V(\vec{r}')\psi(\vec{r}')d^3r'\right|^2$$

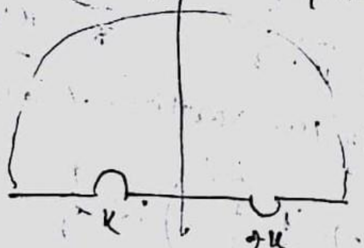
$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4 A^2}\left|\langle \psi_{in}|V|\psi\rangle\right|^2 \qquad A = \frac{1}{(2\pi)^{3/2}}$$

~~The 1st Born Approximation~~ —

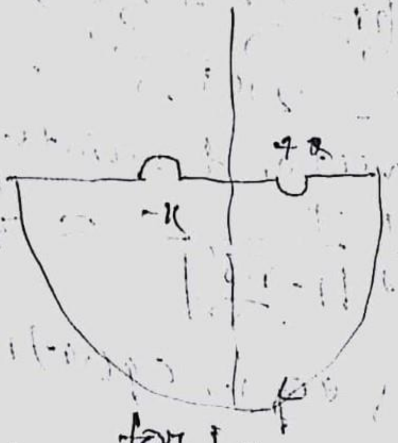
Lipmann Schwinger equation

We have the Green's function for Hammel's function $(\nabla^2 + k^2)$ op. is

$$G(k,\vec{R}) = -\frac{1}{4\pi}\frac{\exp(ik|\vec{R}|)}{|\vec{R}|} \qquad |\vec{R}| = |\vec{r} - \vec{r}'|$$



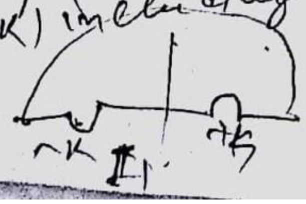
for I_1



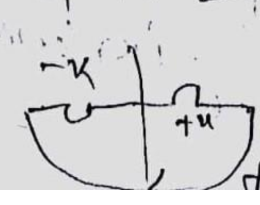
for I_2

We have choose the pole to include and exclude $(-k)$ to find out I_1 & I_2 & finally $G(k,\vec{r},\vec{r}')$ which give us the out going sperical wave $\frac{e^{ikr}}{r}$.

If we choose poles in $(+k)$ excluding & $(-k)$ including for I_1 & I_2.



&



give us $\frac{e^{-ikr}}{r}$

which is incoming wave.

So the complete set of solution

$$\psi_k(\bar{r}) = A e^{i\bar{k}\cdot\bar{r}} - \frac{1}{4\pi}\int \frac{\exp(ik|\bar{r}-\bar{r}'|)}{|\bar{r}-\bar{r}'|} V(\bar{r}')\psi_k(\bar{r}')d^3r' \quad (10)$$

is called the Lipmann - Schwinger equation.

At large distance $\psi_k(\bar{r})$ should satisfy the appropriate boundary condition of the scattering problem.

K has definite magnitude fixed by the energy eigen value but its direction is undetermined.

The choice of the form of $G(\bar{r},\bar{r}')$ is very important to obtain the appropriate form of boundary condition to solve the equation (10)

$$G_{\pm}(\bar{r},\bar{r}') = \frac{\exp(\pm ik|\bar{r}-\bar{r}'|)}{|\bar{r}-\bar{r}'|}$$

In one dimension

$$\psi_k^{\pm}(\bar{r}) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi}\int \frac{e^{\pm ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} V(\bar{r}')\psi_k^{\pm}(\bar{r}')d^3r'$$

$$\left[A = \frac{1}{(2\pi)^{3/2}}\right]$$

The Born Approximation :-

If the potential $V(\vec{r})$ is very weak, it will distort only slightly the incident plane wave. The 1st Born Approximation consists then of approximation corresponds to the 1st iteration of Born series equation-(10) i.e. $\psi(\vec{r}')$ must be the unperturbed incident wave & $\psi(\vec{r})$ become -

$$\psi(\vec{r}) = \psi_{in}(\vec{r}) - \frac{\mu}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} V(\vec{r}')\,\psi_{in}(\vec{r}')\,d^3r'$$

(where $\psi_{in}(\vec{r}')$ is out going plane wave with momentum $\hbar k_0$), is

The scattering amplitude $f(\theta,\phi)$ is

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{k}\cdot\vec{r}'} V(\vec{r}')\, e^{ik_0 r'}\, d^3r'$$

$\psi_{in}(\vec{r}') = e^{ik_0 r'}$ [the incident amplitude A of unit]

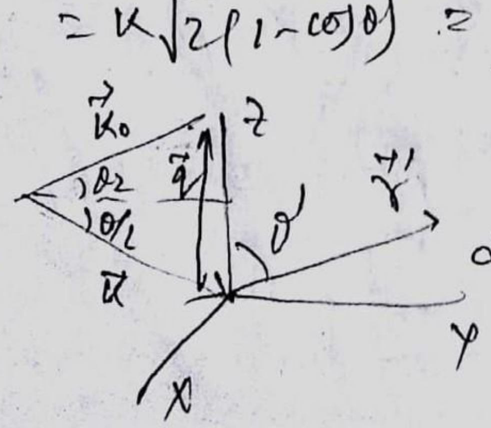
$$\therefore f(\theta\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}'} V(\vec{r}')\,d^3\vec{r}'$$

where $\vec{q} = (\vec{k}_0 - \vec{k})$ and $\hbar\vec{q}$ is momentum transfer, $\hbar\vec{k}_0$ is incident momentum & $\hbar\vec{k}$ is scattered momentum.

for elastic scattering $q = |\vec{k}_0 - \vec{k}|$ and/or $\hbar k_0 = \hbar k$

$$q = \sqrt{k_0^2 + k^2 - 2kk_0\cos\theta} = k\sqrt{2 - 2\cos\theta}$$

$$= k\sqrt{2(1-\cos\theta)} = k\sqrt{2(2\sin^2\theta/2)} = 2k\sin(\theta/2)$$



Let us choose $\vec{q}$ is in z direction and the potential is central $V(\vec{r}') = V(r')$

$$\therefore \vec{q}\cdot\vec{r}' = q\, r'\cos\theta'$$

$$\therefore \int e^{i\vec{q}\cdot\vec{r}'} V(\vec{r}')\, d^3r'$$

$$= \int_0^\infty r'^2 V(\vec{r}')\, dr' \int_0^\pi e^{iqr'\cos\theta'} \sin\theta' d\theta' \int_0^{2\pi} d\phi'$$

$$= 2\pi \int_0^\infty r'^2 V(r') dr' \int_{-1}^{1} e^{iqr'x} dx \quad [x = \cos\theta']$$

$$= 2\pi \int_0^\infty r' V(r') dr' \left(\frac{e^{iqr'} - e^{-iqr'}}{q} \right)$$

$$= \frac{4\pi}{q} \int_0^\infty r' V(r') dr' \sin(qr')$$

$$\therefore \boxed{f(\theta) = -\frac{2\mu}{\hbar^2 q} \int_0^\infty r' V(r') \sin(qr') dr'} \quad \text{--- (11)}$$

Validity of 1st Born Approximation

(1) 1st Born approximation indicate there is only one scattering (No multiple scattering for a particle)

It is valid whenever the wave function $\psi(\vec{r})$ is only slightly different from the incident plane wave i.e the 2nd term is very small compared to the 1st.

$$\left| \frac{\mu}{2\pi\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} V(\vec{r}') e^{i\vec{k}_0\cdot\vec{r}'} d^3r' \right|^2 \ll |\psi_{inc}(\vec{r})|^2$$

Let $\psi_{inc}(\vec{r}) = e^{i\vec{k}_0\cdot\vec{r}}$ $\therefore |\psi_{inc}(\vec{r})|^2 = 1$

for azymuthally semetric & $k = k_0$

$$\left| \frac{\mu}{\hbar^2} \int_0^\infty r' e^{ik\cdot r'} V(r') dr' \int e^{ikr'\cos\theta'} \sin\theta' d\theta' \right| \ll 1$$

$$\frac{\mu}{\hbar^2 k} \left| \int_0^\infty V(r') (e^{2ikr'} - 1) dr' \right| \ll 1$$

So this should be valid if $k \gg 1$ i.e incident energy is high enough with respect to the very weak scattering potentials.

Application of Born approximation

(1) Low energy Soft sphere scattering

$$V(\vec{r}) = V_0 \quad \text{if } r \leq a$$
$$= 0 \quad \text{if } r > a$$

We have $$f(\theta,\phi) \simeq -\frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}}\, V(r)\, d^3r$$
$$\simeq -\frac{2m}{\hbar^2}\int_0^\infty \frac{\sin(qr)}{q}\cdot V(r)\, r\, dr \qquad \text{eq-(11)}$$

$\hbar q$ is momentum change.

For low energy (qr) is very small.

then $\sin(qr) \to qr$

$$\therefore f(\theta,\phi) = -\frac{2m}{\hbar^2}\int_0^a \frac{qr}{q}\, r\, V(r)\, dr$$
$$= -\frac{2m}{\hbar^2}\int_0^a r^2 V_0\, dr = -\frac{2mV_0}{\hbar^2}\cdot\frac{a^3}{3}$$

$$\therefore \left|\frac{d\sigma}{d\Omega}\right| = \frac{4m^2 V_0^2 a^6}{9\hbar^4}$$

$\therefore$ Total scattering cross section

$$\sigma = \int \left|\frac{d\sigma}{d\Omega}\right| d\Omega = 4\pi \times \frac{4m^2V_0^2a^6}{9\hbar^4}$$
$$= \frac{16\pi}{9}\frac{m^2V_0^2a^6}{\hbar^4}$$

Calculate σ for Yukawa potential $V(r) = \beta \frac{e^{-\alpha r}}{r}$

$$f(\theta,\phi) = -\frac{\mu}{2\pi\hbar^2}\int e^{-i\vec{q}\cdot\vec{r}}\, \beta\, \frac{e^{-\alpha r}}{r}\, d^3r$$

$$= -\frac{2\mu\beta}{\hbar^2 q}\int_0^\infty \frac{e^{-\alpha r}}{r}\cdot\frac{\sin(qr)}{r}\cdot r^2\, dr$$

$$= -\frac{2\mu\beta}{i\hbar^2 q}\int_0^\infty e^{-\alpha r}\left(e^{iqr} - e^{-iqr}\right) dr$$

$$= \frac{i2\mu\beta}{\hbar^2 q}\int_0^\infty \left[e^{(iq-\alpha)r} - e^{-(iq+\alpha)r}\right] dr$$

$$\left[\int_0^\infty x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}\right]$$

$$= \frac{i2\mu\beta}{\hbar^2 q}\int_0^\infty \left[e^{-(-iq+\alpha)r}\, r^0 - e^{-(\alpha+iq)r}\, r^0\right] dr$$

$$= \frac{i2\mu\beta}{\hbar^2 q}\left[\frac{0!}{(\alpha-iq)} - \frac{0!}{\alpha+iq}\right]$$

$$= \frac{i2\mu\beta}{\hbar^2 q}\left[\frac{\alpha+iq-\alpha+iq}{\alpha^2+q^2}\right]$$

$$= -\frac{4\mu\beta q}{\hbar^2 q(\alpha^2+q^2)}$$

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2 = \frac{16\mu^2\beta^2}{\hbar^4(\alpha^2+q^2)^2}$$

$$\therefore \sigma = \frac{64\pi\mu^2\beta^2}{\hbar^4(\alpha^2+q^2)^2}$$

Scattering by a Gaussian potential

$$V(r) = -V_0 e^{-\frac{r^2}{a^2}}$$

$$f(\theta,\phi) = -\frac{m \cdot 2\pi \cdot 2}{\hbar^2 \cdot 2\pi} \int_0^\infty \frac{\sin qr}{(qr)} \cdot V(r)\, r^2 dr$$

$$= -\frac{2m}{\hbar^2 q} \int_0^\infty \left(\frac{e^{iqr} - e^{-iqr}}{2i}\right)\left(-V_0 e^{-\frac{r^2}{a^2}}\right) r\,dr$$

$$= \frac{-imV_0}{\hbar^2 q} \int_0^\infty \left[e^{iqr - \frac{r^2}{a^2}} - e^{-iqr - \frac{r^2}{a^2}}\right) r\,dr$$

$$\left[iqr - \frac{r^2}{a^2}\right.$$

$$= -\left[\frac{r^2}{a^2} - 2iqr\right]$$

$$= -\left[\frac{r^2}{a^2} - 2\cdot\frac{r}{a}\left(\frac{iqa}{2}\right) + \left(\frac{iqa}{2}\right)^2 - \left(\frac{iqa}{2}\right)^2\right]$$

$$= -\left(\frac{r}{a} - \frac{iqa}{2}\right)^2 + \left(\frac{iqa}{2}\right)^2$$

$$-\left[iqra + \frac{r^2}{a^2}\right) = -\left[\left(\frac{r}{a} + \frac{iqa}{2}\right)^2 + \left(\frac{iqa}{2}\right)^2\right]$$

$$\therefore f(\theta,\phi) = -\frac{imV_0}{2\hbar^2 q} \int_{-\infty}^{\infty} \left[\exp\left[-\left(\frac{r}{a} - \frac{iqa}{2}\right)^2\right] \cdot e^{\left(\frac{iqa}{2}\right)^2} - \exp\left[-\left(\frac{r}{a} + \frac{iqa}{2}\right)^2\right] \cdot e^{\left(\frac{iqa}{2}\right)^2}\right] r\,dr$$

$$= \frac{-imV_0}{2\hbar^2 q} e^{-\frac{q^2a^2}{4}} \left[\int_{-\infty}^{\infty} e^{-\left(\frac{r}{a} - \frac{iqa}{2}\right)^2} r\,dr - \int_{-\infty}^{\infty} e^{-\left(\frac{r}{a} + \frac{iqa}{2}\right)^2} r\,dr\right]$$

Let $\left(\frac{r}{a} - \frac{iqa}{2}\right) = t$

$\therefore dr = a\,dt$

$r = \left(t + \frac{iqa}{2}\right) a$

$$\int_{-\infty}^{\infty} e^{-t^2}\left(\frac{t}{a}+\frac{iqa}{2a}\right) a^2 dt$$

$$= a^2\left[\int_{-\infty}^{\infty} e^{-t^2}\cdot t\,dt + \int_{-\infty}^{\infty} e^{-t^2}\left(\frac{iqa}{2}\right)dt\right]$$

$\quad\quad\quad\quad \hookrightarrow 0$

$$= \frac{iqa^3}{2}\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-\left(\frac{r}{a}+\frac{iqa}{2}\right)^2} r\,dr = \int_{-\infty}^{\infty} e^{-t^2}\left(\frac{t}{a}-\frac{iqa}{2a}\right) a^2 dt$$

$$= a^2\left[-\int_{-\infty}^{\infty} e^{-t^2}\left(\frac{iqa}{2}\right)dt\right] = -\left(\frac{iqa^3}{2}\right)\sqrt{\pi}$$

$\quad\quad \hookrightarrow 0$

$$\therefore f(\theta,\phi) = -\frac{imV_0a^2}{2\hbar^2 q} e^{-\frac{q^2a^2}{4}}\cdot 2\sqrt{\pi}\left(\frac{iqa}{2}\right)$$

$$= \frac{mV_0\sqrt{\pi}a^3}{2\hbar^2} e^{-\frac{q^2}{4}}$$

$$\therefore \boxed{\sigma = 4\pi\cdot\frac{m^2V_0^2\pi a^6}{4\hbar^4}\cdot e^{-\frac{q^2a^2}{2}}}$$

$$= \frac{m^2V_0^2\pi^2a^6}{\hbar^4}\cdot e^{-\frac{q^2a^2}{2}}$$

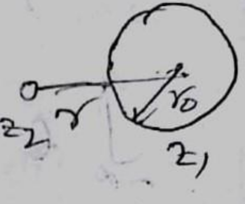
$q = 2k\sin\theta/2$ [θ is angle between k & k' i.e angle between incident & scattered propagation vector]

$$\sigma = \frac{m^2V_0^2\pi^2a^6}{\hbar^2}\, e^{-2k^2a^2\sin^2(\theta/2)}$$

Scattering by a screened coulomb potential, Rutherford's scattering formula from Born approximation:

Coulomb potential for a nucleus of atomic no. Z_1 and the scattered particle Z_2 is $V(r) = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r} \cdot e^{-r/r_0}$ (SI system)

where r_0 is called screening radius



From 1st Born approximation -

$$f(\theta) = -\frac{m}{2\pi\hbar^2} \int e^{i\vec{q}\cdot\vec{r}} V(r)\, r^2 d^3r$$

$$\simeq -\frac{2m}{\hbar^2 q} \int_0^\infty \sin(qr)\, V(r)\, r\, dr$$

$$\simeq -\frac{2m}{\hbar^2 q} \int_0^\infty \sin(qr) \left(\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0}\right) \cdot e^{-\frac{r}{r_0}} dr$$

$$\simeq -\frac{2m Z_1 Z_2 e^2}{4\pi\epsilon_0 \hbar^2 q} \cdot \frac{1}{2i} \int_0^\infty \left[e^{\left(iqr - \frac{r}{r_0}\right)} - e^{\left(-iqr - \frac{r}{r_0}\right)} \right] dr$$

$$\simeq \frac{im Z_1 Z_2 e^2}{4\pi\epsilon_0 \hbar^2 q} \left[\int_0^\infty e^{-\left(\frac{1}{r_0} - iq\right)r} dr - \int_0^\infty e^{-r\left(\frac{1}{r_0} + iq\right)} dr \right]$$

$$= \frac{im Z_1 Z_2 e^2}{4\pi\epsilon_0 \hbar^2 q} \left[\frac{0!}{\left(\frac{1}{r_0} - iq\right)} - \frac{0!}{\left(\frac{1}{r_0} + iq\right)} \right] \qquad \left[\int_0^\infty e^{-\alpha r} r^n dr = \frac{n!}{\alpha^{n+1}} \right]$$

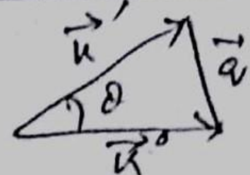
$$= \frac{im Z_1 Z_2 e^2}{4\pi\epsilon_0 \hbar^2 q} \left[\frac{\frac{1}{r_0} + iq - \frac{1}{r_0} + iq}{\left(\frac{1}{r_0}\right)^2 + q^2} \right]$$

$$= -\frac{2m Z_1 Z_2 e^2}{4\pi\epsilon_0 \hbar^2 \left(q^2 + \frac{1}{r_0^2}\right)}$$

$$\sigma = 4\pi |f(\theta)|^2 = 4\pi \left[\frac{2m Z_1 Z_2 e^2}{4\pi\epsilon_0 \hbar^2 \left(q^2 + \frac{1}{r_0^2}\right)} \right]^2$$

• q may be replaced by

$q = 2k\sin\theta/2$



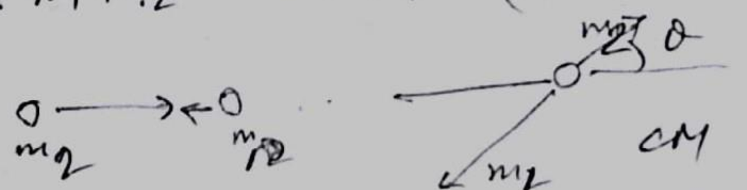
$$\therefore \sigma = 4\pi \left(\frac{2m z_1 z_2 e^2}{4\pi\epsilon_0 \hbar^2} \right)^2 \left(\frac{1}{4k^2 \sin^2\frac{\theta}{2} + \frac{1}{r_0^2}} \right)^2$$

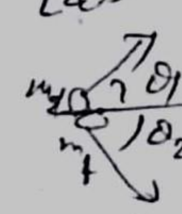
A quantitative idea

For Rutherford scattering —

$z_1 \approx 79$ $(A_1 = 197)$ & $z_2 = 2$ $(A_2 = 4)$

~~$\frac{m_1 m_2}{m_1+m_2}$~~ $\frac{m_2}{m_1} \approx \left(\frac{4}{197}\right)$





The scattering angle in CM system is $\theta \simeq 60^\circ$

then in Lab frame is θ_1, $\tan\theta_1 \approx \frac{\sin\theta}{\cos\theta + m_2/m_1}$. Let $\theta = 60^\circ$, $\tan\theta_1 = \frac{\sqrt{3}/2}{\frac{1}{2} + \left(\frac{4}{197}\right)}$

$\theta_1 \simeq 61^\circ$ Energy $E = \frac{\hbar^2 k^2}{2\mu}$

$$\frac{d\sigma}{d\Omega} = \left(\frac{2\mu z_1 z_2 e^2}{4\pi\epsilon_0 \hbar^2} \right)^2 \frac{1}{4k^2 \sin^2\left(\frac{\theta_1}{2}\right)}$$

$\frac{1}{r_0^2}$ is negli as $r_0 \to \infty$ for ordinary coulomb potential

$$= \left(\frac{2\mu}{\hbar^2 k^2}\right)^2 \left(\frac{z_1 z_2}{4\pi\epsilon_0}\right)^2 \left(\frac{e^2}{\hbar c}\right)^2 (\hbar c k^2)^2 \cdot \frac{1}{4k^2 \sin^2\left(\frac{\theta_1}{2}\right)}$$

$$= \left(\frac{1}{E}\right)^2 \left(\frac{z_1 z_2}{4\pi\epsilon_0}\right)^2 \alpha^2 (\hbar c k^2)^2 \cdot \frac{1}{4k^2 \sin^2(30.5^\circ)}$$

$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$ $z_1 = 79$ $z_2 = 2$ $\frac{1}{4\pi\epsilon_0} = 9\times10^9$ $\hbar c \simeq 197.3$ MeV

putting all value

$\frac{d\sigma}{d\Omega} = 0.31 \times 10^{-28}\, m^2 = 0.31$ barn.