VIVEKANANDA COLLEGE THAKURPUKUR KOLKATA-700063

NAAC ACCREDITED ‘A’ GRADE

Topic: Plucked String and Stuck String

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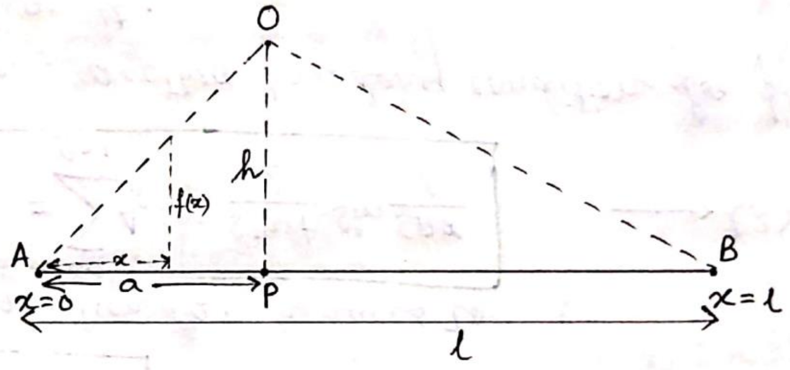
Plucked String.

When a string is firmly fixed with tension at its two ends and a point of string is plucked at a certain distance (say 'a') from one end to a small height (say 'h'), so that the string makes two sides of a traingle. If it is then released, it will begin vibrating, and hence the string is said to be vibrating by plucking.



The differential equation of a vibrating string is,

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad \text{where } v = \sqrt{\frac{T}{\mu}} \quad \longrightarrow (1)$$

[$T \rightarrow$ Tension, $\mu \rightarrow$ linear mass density.]

The general solution for a string of length 'l' will be

$$y = \sum_{s=1}^{\infty} \left\{ \left(A_s \cos \frac{s\pi v t}{l} + B_s \sin \frac{s\pi v t}{l} \right) \sin \frac{s\pi x}{l} \right\} \longrightarrow (2)$$

Now, in case of plucked string, the boundary condition is,

$$\boxed{\frac{\partial y}{\partial t} = 0 \quad \text{when } t=0}.$$

From (2); $\frac{\partial y}{\partial t} = \sum_{s=1}^{\infty}\left(-\frac{s\pi v}{l} A_s \sin\frac{s\pi v t}{l} + \frac{s v \pi}{l} B_s \cos\frac{s\pi v t}{l}\right)\sin\frac{s\pi x}{l}$

$$\Rightarrow \quad 0 = \sum_{s=1}^{\infty} B_s \frac{s\pi v}{l}\sin\frac{s\pi x}{l}$$

$$\therefore \boxed{B_s = 0}.$$

So, equation (2) reduces to

$$\boxed{y = \sum_{s=1}^{\infty} A_s \cos\frac{s\pi v t}{l}\sin\frac{s\pi x}{l}} \longrightarrow (3)$$

Again, another boundary condition for plucked string is,

$\boxed{y = f(x) \quad \text{at } t=0}$ [displacement is position dependent as shown in figure].

So, from equation (3),

$$f(x) = \sum_{s=1}^{\infty} A_s \sin\frac{s\pi x}{l} \longrightarrow (4)$$

To find out the value of A_s by Fourier's method, we shall multiply $\sin\frac{m\pi x}{l}$ (m is integer) on bothsides of equation (4) and integrate from $x=0$ to $x=l$,

$$\int_0^l f(x)\sin\frac{m\pi x}{l}\,dx = A_s\int_0^l \sin\frac{s\pi x}{l}\sin\frac{m\pi x}{l}\,dx$$

$$= A_s.\frac{l}{2} \quad \text{when } s=m \text{ or}$$

zero otherwise.

$$\therefore \boxed{A_s = \frac{2}{l}\int_0^l f(x)\sin\frac{s\pi x}{l}\,dx} \longrightarrow (5) \qquad \because s=m.$$

Now, for 'AP' portion of string;

$$\frac{f(x)}{x} = \frac{h}{a}.$$

$$\therefore f(x) = \frac{h}{a}x \text{ for } 0<x<a$$

And for PB portion of string;

$$\frac{f(x)}{l-x} = \frac{h}{l-a}.$$

$$\therefore f(x) = \frac{h(l-x)}{(l-a)}$$

for $a<x<l$

So, from equation (5),

$$A_s = \frac{2}{l}\left[\frac{h}{a}\int_0^a x\sin\frac{s\pi x}{l}\,dx + \frac{h}{l-a}\int_a^l (l-x)\sin\frac{s\pi x}{l}\,dx\right].$$

$$\text{or, } A_s = \frac{2h}{l}\left[\frac{1}{a}\int_0^a x\sin\frac{s\pi x}{l}\,dx + \frac{l}{l-a}\int_a^l \sin\frac{s\pi x}{l}\,dx - \frac{1}{l-a}\int_a^l x\sin\frac{s\pi x}{l}\,dx\right].$$

$$\text{or, } A_s = \frac{2h}{l}\left[\frac{1}{a}I_1 + \frac{l}{(l-a)}I_2 - \frac{1}{l-a}I_3\right]. \longrightarrow (6)$$

Now,

$$\int x \sin\frac{s\pi x}{l}\,dx = -x\left(\frac{l}{s\pi}\right)\cos\frac{s\pi x}{l} - \int\left(-\frac{l}{s\pi}\right)\cos\frac{s\pi x}{l}\,dx$$

$$= -x\left(\frac{l}{s\pi}\right)\cos\frac{s\pi x}{l} + \left(\frac{l}{s\pi}\right)^2\sin\frac{s\pi x}{l}$$

$$I_1 = -\frac{la}{s\pi}\cos\frac{s\pi a}{l} + \left(\frac{l}{s\pi}\right)^2\sin\frac{s\pi a}{l}$$

$$I_3 = -\frac{l^2}{s\pi}\cos s\pi + \frac{l^2}{s\pi}\sin s\pi - \left[-\frac{la}{s\pi}\cos\frac{s\pi a}{l} + \left(\frac{l}{s\pi}\right)^2\sin\frac{s\pi a}{l}\right]$$

$$I_2 = -\left[\frac{l}{s\pi}\cos\frac{s\pi x}{l}\right]_a^l$$

$$= \left[\frac{l}{s\pi}\cos\frac{s\pi x}{l}\right]_l^a$$

$$= \frac{l}{s\pi}\left(\cos\frac{s\pi a}{l} - \cos s\pi\right).$$

Substituting the values of I_1, I_2 & I_3 in equation (6) and on simplification we will get,

$$\boxed{A_s = \frac{2hl^2}{(s\pi)^2 a(l-a)}\sin\frac{s\pi a}{l}} \longrightarrow (7)$$

Solution for a plucked string becomes, [using (3) & (7)]

$$\boxed{y = \sum_{s=1}^{\infty}\frac{2hl^2}{(s\pi)^2 a(l-a)}\sin\frac{s\pi a}{l}\cos\frac{s\pi vt}{l}\sin\frac{s\pi x}{l}}$$

It is clear from the above expression, the coefficients A_s vary inversely with s^2, therefore, the amplitudes of the higher harmonic fall off rapidly.

Note:

If the string is plucked at the midpoint i.e. '$a = \frac{l}{2}$', then '$\sin \frac{s\pi a}{l}$' becomes '$\sin \frac{s\pi}{2}$'. For even 's', $\sin \frac{s\pi}{2} = 0$, i.e. the even harmonics will vanish. All these harmonics have a node at $a = l/2$.

The amplitude of vibration for fundamental i.e. $s = 1$

$$A_1 = \frac{2hl^2}{a(l-a)\pi^2} \sin \frac{\pi a}{l}.$$

Whereas, that for the third harmonic i.e. $s = 3$.

$$A_3 = \frac{2hl^2}{9a(l-a)\pi^2} \sin \frac{3\pi a}{l}.$$

and so on.

Struck String.

If a string stretched between two fixed supports at its ends is struck at a point by a hammer, a sudden impulse is imparted to the point of striking but the velocity of all other points on the string remains initially zero.

The duration of contact between striking hammer and string is assumed to be very small, so that the impact can be considered to have ceased before the disturbance can move over to the other regions of the length.

Thus the initial velocity is confined to a very short length.

Let, 'l' be the length of the stretched string supported at $x=0$ and $x=l$. It is struck at an infinitesimally small region from $x=a$ to $x=a+dx$ and this region attains velocity 'u' at $t=0$.

We know, the expression for displacement of stretched string;

$$y = \sum_{s=1}^{\infty}\left(A_s \cos\frac{s\pi vt}{l} + B_s \sin\frac{s\pi vt}{l}\right)\sin\frac{s\pi x}{l} \qquad \text{——— (1)}$$

The Boundary condition is

i) $y = 0$ everywhere at $t = 0$. ie. initial displacement is zero.

By imposing this condition on equation (1), we get $A_s = 0$.

Hence, $\boxed{y = \sum_{s=1}^{\infty} B_s \sin\frac{s\pi v t}{l} \sin\frac{s\pi x}{l}}$ $\longrightarrow$ (2).

From equation (2),

The velocity $\frac{\partial y}{\partial t} = \frac{\pi v}{l} \sum_{s=1}^{\infty} s.B_s \sin\frac{s\pi x}{l} \cos\frac{s\pi v t}{l}$

The initial velocity at any point 'x',

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \frac{\pi v}{l} \sum_{s=1}^{\infty} s B_s \sin\frac{s\pi x}{l}.$$

or, $$\dot{y}_0 = \frac{\pi v}{l} \sum_{s=1}^{\infty} s B_s \sin\frac{s\pi x}{l}$$

To find out B_s by Fourier's method,

$$\int_0^l \dot{y}_0 \sin\frac{m\pi x}{l}\, dx = \int_0^l \frac{\pi v s}{l} B_s \sin\frac{s\pi x}{l} \sin\frac{m\pi x}{l}\, dx$$

$$= \frac{\pi v s}{l} . B_s . \frac{l}{2} = \frac{s\pi v B_s}{2}. \quad \left[\text{for } s = m \text{ only}\right]$$

or zero otherwise.

Now, $\dot{y}_0 = 0$ for all values of x except for the region from $x=a$ to $x=a+dx$, where

$\dot{y}_0 = u$.

$$\therefore \int_a^{a+dx} u \sin\frac{s\pi a}{l}\, dx = \frac{s\pi v B_s}{2}$$

or, $\dfrac{s\pi v B_s}{2} = C \sin\dfrac{s\pi a}{l}$ where $C = \displaystyle\int_a^{a+dx} u\,dx$

$$\therefore \boxed{B_s = \frac{2C}{s\pi v} \sin\frac{s\pi a}{l}} \longrightarrow (3).$$

From (2);

$$\therefore \boxed{y = \frac{2C}{\pi v} \sum_{s=1}^{\infty} \frac{1}{s} \sin\frac{s\pi a}{l} \sin\frac{s\pi x}{l} \sin\frac{s\pi v t}{l}}$$

<u>Note</u>: The amplitudes of the harmonics fall with 's' as it is inversely proportional to 's'

<u>Comparison</u>:

1. For plucked string, it has initial displacement with zero initial velocity, whereas for struck string, its displacement is zero but it has some initial velocity at the region of striking.
2. The amplitudes of harmonics falls more rapidly in case of plucked string.