VIVEKANANDA COLLEGE THAKURPUKUR KOLKATA-700063

NAAC ACCREDITED ‘A’ GRADE

Topic: Transverse Vibration of Stretched String

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Plane Progressive (Travelling) Waves.

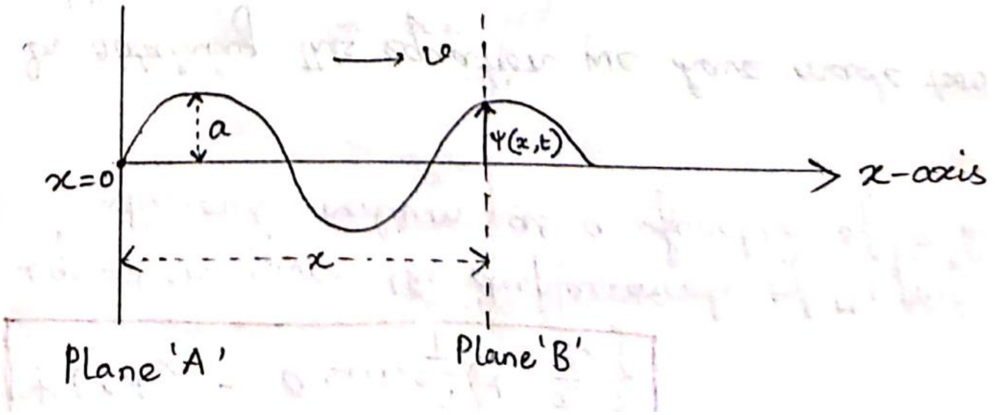
Consider a harmonic wave travelling in a medium with a velocity v. Consider two planes A and B separated by a distance 'x'. Let us assume, that the wave is created at $x=0$ (plane A). So, the displacement of the particle at plane A is,

$$\psi(0,t) = a \sin \frac{2\pi}{T} t \quad ; \text{ where } 'a' \rightarrow \text{amplitude}, \ 'T' \rightarrow \text{Time period}.$$

Now, the wave, which creates at 'A' will reach 'B' in $\frac{x}{v}$ seconds [since 'v' is velocity of wave].

So, the displacement of particles at 'B' must be same as the displacement of particles at 'A' had '$\frac{x}{v}$' seconds earlier.

$\therefore$ Displacement of particles at 'B' at time 't'
= Displacement of particles at 'A' at time '$(t-\frac{x}{v})$'.



$$\therefore \boxed{\psi(x,t) = a \sin\left\{\frac{2\pi}{T}\left(t - \frac{x}{v}\right)\right\}}$$

This equation gives the displacements of the particles of the continuous medium as a function of 'x' & 't'.

N.B. In obtaining this equation we have made two assumptions.

(i) The amplitude of particle oscillations does not change during wave propagation and

(ii) The medium is isotropic and homogeneous, so, the wave velocity does not change.

Now, By definition of wavelength (λ), two particles separated by a distance 'λ' are in the same state of motion. i.e. spatial periodicity of wave.

$$\psi(x+\lambda, t) = a \sin\left\{\frac{2\pi}{T}\left(t - \frac{x+\lambda}{v}\right)\right\}.$$

$$= a \sin\left\{\frac{2\pi}{T}\left(t - \frac{x}{v}\right) - \frac{2\pi}{T}\cdot\frac{\lambda}{v}\right\}$$

$$= a \sin\left\{\frac{2\pi}{T}\left(t - \frac{x}{v}\right) - 2\pi\right\} \quad \left[\because \lambda = \frac{v}{T}\right] \quad \lambda = vT$$

$$= a \sin\left\{\frac{2\pi}{T}\left(t - \frac{x}{v}\right)\right\}$$

$$= \psi(x,t)$$ i.e. Same displacement.

The function $\psi(x,t)$ repeats itself in a distance 'λ'.

The Classical Wave Equation

We have,

$$\psi(x,t) = a\sin\left\{\frac{2\pi}{T}\left(t-\frac{x}{v}\right)\right\}.$$

$$= a\sin\left\{\frac{2\pi}{vT}(vt-x)\right\}.$$

$$= a\sin\left\{\frac{2\pi}{\lambda}(vt-x)\right\}.$$

Differentiating twice w.r.t 't', keeping 'x' constant.

$$\frac{\partial^2\psi}{\partial t^2} = -a\left(\frac{2\pi v}{\lambda}\right)^2 \sin\left\{\frac{2\pi}{\lambda}(vt-x)\right\}.$$

$$= -\left(\frac{2\pi v}{\lambda}\right)^2 \psi$$

Again, differentiating ψ w.r.t 'x' twice, keeping 't' constant

$$\frac{\partial^2\psi}{\partial x^2} = -a\left(\frac{2\pi}{\lambda}\right)^2 \sin\left\{\frac{2\pi}{\lambda}(vt-x)\right\}$$

$$= -\left(\frac{2\pi}{\lambda}\right)^2 \psi$$

Comparing these two we get,

$$\boxed{\frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial x^2}}$$

here 'v' is the wave velocity. (also called phase velocity

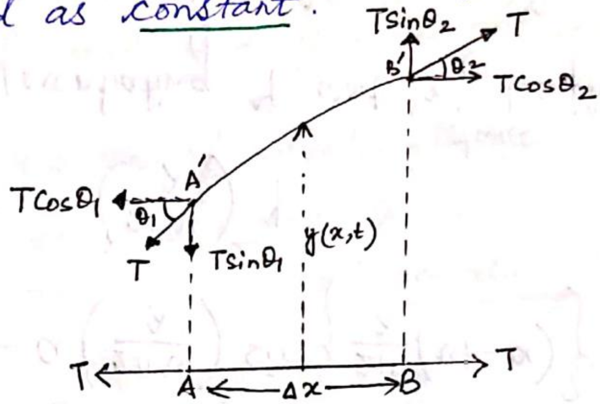
Transverse Vibration of String

Consider a uniform stretched string of tension T lying along the x-axis in equillibrium position. The string is divided into a large number of infinitesimally small elements each of length Δx.

The theory is based on two assumptions:

i) The stiffness (cohesive force) of the string is very high, so that longitudinal vibration is neglected.

ii) The amplitude of transverse vibration should be small, so that the tension of the string can be treated as constant.



The element is displaced along the y-axis, so the element is no longer exactly straight, gets a slight curvature and the tension is tangential.

Let, $y(x,t)$ be the transverse displacement of the element located at x at time 't'. When the string is released, the displacement of element will change with time as well as with position (as shown in fig).

For $\Delta x \to 0$, the net forces on the element along x and y directions are.

$$\boxed{\begin{aligned} F_x &= T\cos\theta_2 - T\cos\theta_1 \\ F_y &= T\sin\theta_2 - T\sin\theta_1 \end{aligned}}$$

The transverse component of tension on the element is

$$F_y = T(\sin\theta_2 - \sin\theta_1)$$

$$= T(\tan\theta_2 - \tan\theta_1)$$

$$= T\left\{ \frac{\partial y}{\partial x}\Big|_{x=x_2} - \frac{\partial y}{\partial x}\Big|_{x=x_1} \right\}$$

$$= T\{ f(x_2) - f(x_1) \}$$

[Assuming the displacement (transverse) is very small, so that θ_1 & θ_2 are small angles

$\cos\theta_1 \simeq 1 \quad \cos\theta_2 \simeq 1$

$\sin\theta_1 \simeq \tan\theta_1 \quad \sin\theta_2 \simeq \tan\theta_2$]

Here only the transverse component of force provides the necessary restoring force for sustaining the vibration.

Where $f(x)$ stands for $\frac{\partial y}{\partial x}$.

Using Taylor's series expansion

$$f(x_2) = f(x_1) + (x_2 - x_1)\frac{\partial f}{\partial x}\Big|_{x=x_1}$$

[Neglecting higher order terms]

$$= f(x_1) + \Delta x . \frac{\partial f}{\partial x}\Big|_{x=x_1}.$$

$\because (x_2 - x_1) = \Delta x \downarrow$ very small

$$\therefore f(x_2) - f(x_1) = \Delta x . \frac{\partial f}{\partial x}\Big|_{x=x_1}$$

$$= \Delta x . \frac{\partial^2 y}{\partial x^2}$$

So, we get, $F_y = T\Delta x \frac{\partial^2 y}{\partial x^2}$

An ideal string is linear distribution of mass points.

Let, 'μ' be the mass per unit length of the string.

So, Newton's force $= \mu \Delta x \frac{\partial^2 y}{\partial t^2}$

[Where
$\mu\Delta x$ = mass of the element

$\frac{\partial^2 y}{\partial t^2}$ = acceleration

$\because y \rightarrow$ displacement]

For dynamic equillibrium,

$$\mu \Delta x \frac{\partial^2 y}{\partial t^2} = T\Delta x \frac{\partial^2 y}{\partial x^2}$$

$$\therefore \boxed{\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}}$$

Comparing the above equation with classical wave equation we get,

$$\boxed{v = \sqrt{\frac{T}{\mu}}} \rightarrow$$ velocity of transverse wave on a string.

From the above equation it is clear that, the displacement y is a function of both x and 't'.

So, let us assume that, the solution of the differential equation may be like

$$\boxed{y(x,t) = f(x)g(t)}$$

$$\therefore \frac{\partial^2 y}{\partial t^2} = f(x)\frac{d^2 g}{dt^2} \quad \& \quad \frac{\partial^2 y}{\partial x^2} = g(t)\frac{d^2 f}{dx^2}$$

So, $f(x)\frac{d^2g}{dt^2} = v^2 g(t)\frac{d^2f}{dx^2}$ [where $v^2 = \frac{T}{\mu}$]

or, $\frac{1}{g(t)}\frac{d^2g}{dt^2} = \frac{v^2}{f(x)}\frac{d^2f}{dx^2}$

It is seen that, L.H.S is a function of time only whereas R.H.S is a function of position only and they are equal. It is only possible when bothsides of the above equation are constants.

Since the solution must be oscillatory in nature we can take

$$\frac{1}{g}\frac{d^2g}{dt^2} = \frac{v^2}{f}\frac{d^2f}{dx^2} = -\omega^2$$

So, $\frac{d^2g}{dt^2} + \omega^2 g = 0$ and $\frac{d^2f}{dx^2} + \frac{\omega^2 f}{v^2} = 0$

$\therefore g(t) = A\cos\omega t + B\sin\omega t$; $f(x) = C\cos\frac{\omega x}{v} + D\sin\frac{\omega x}{v}$

where A, B, C and D are all arbitrary constants.

$$\therefore \boxed{y(x,t) = \left(C\cos\frac{\omega x}{v} + D\sin\frac{\omega x}{v}\right)\left(A\cos\omega t + B\sin\omega t\right)}$$

Now the constants can be evaluated from boundary conditions.

i) $y = 0$ when $x = 0$ for all values of time 't'.

ii) $y = 0$ when $x = l$ for all values of 't'.

The string of length 'l' is supposed to be fixed at both ends.

from boundary condition<i> implemented on the expression of 'y', we get,

$$0 = C\,(A\cos\omega t + B\sin\omega t).$$

Which implies $C = 0$.

$$\therefore\ y = D\sin\frac{\omega x}{v}\,(A\cos\omega t + B\sin\omega t).$$

Again from boundary condition <ii>

$$0 = D\sin\frac{\omega l}{v}\,(A\cos\omega t + B\sin\omega t).$$

since $D \neq 0$ (non trivial solution).

$$\sin\frac{\omega l}{v} = 0.$$

$$\therefore\ \frac{\omega l}{v} = s\pi$$ where 's' is integer.
$s = 1, 2, 3, \ldots$

From this, it is clear that the solution exists only when the above condition satisfies.

Thus, the vibration of the string will exists only for particular frequencies obeying the above condition. These frequencies are known as normal modes of frequencies.
or eigen frequencies

$$\therefore\ \boxed{\omega = \frac{s\pi v}{l}}$$

$\therefore y = D\sin\frac{s\pi x}{l}\left(A\cos\frac{s\pi v}{l}t + B\sin\frac{s\pi v}{l}t\right).$

Therefore, The s^{th} mode solution of vibration will be like, (absorbing the constant D)

$$\boxed{y_s = \left(A_s\cos\frac{s\pi vt}{l} + B_s\sin\frac{s\pi vt}{l}\right)\sin\frac{s\pi x}{l}}$$

General sol^n:

$$\therefore y(x,t) = \sum_{s=1}^{\infty} y_s$$

$$\therefore y(x,t) = \sum_{s=1}^{\infty}\left\{\left(A_s\cos\frac{s\pi vt}{l} + B_s\sin\frac{s\pi vt}{l}\right)\sin\frac{s\pi x}{l}\right\}.$$

Now, $\omega = \frac{s\pi v}{l}.$

or, $2\pi f = \frac{s\pi v}{l}.$

$\therefore f = \frac{s\pi}{2\pi}\cdot\frac{v}{l} = \frac{s}{2l}\cdot v = \frac{s}{2l}\sqrt{\frac{T}{\mu}}.$

for $s=1$ $\boxed{f = \frac{1}{2\pi}\sqrt{\frac{T}{\mu}}.}$ (fundamental frequency).

$s=2 \Rightarrow$ second harmonic (octave)

$s=3 \Rightarrow$ third harmonic and so on.