VIVEKANANDA COLLEGE THAKURPUKUR KOLKATA-700063

NAAC ACCREDITED 'A' GRADE

Topic: Superposition of Harmonic Oscillations

Course Title: Wave and Optics

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Unit:

Semester: 2

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Name of the Department: Physics

2. Superposition of Harmonic Oscillations:

Principle of superposition:

"The resultant of two or more harmonic displacements is simply the algebric sum of the individual displacements."

[This holds only if the equation of motion is linear]

(a) Superposition of two collinear harmonic oscillations having equal frequencies.

Let, we have two SHMs of equal frequencies but of different amplitudes and phase constants acting on a particle (or a system) in the x- direction.

The displacements x_1 and x_2 of the two harmonic motions having same angular frequency 'ω' are given by

$$x_1 = a\cos(\omega t + \phi_1).$$
$$x_2 = b\cos(\omega t + \phi_2).$$

where, a, b are the amplitudes. ϕ_1, ϕ_2 are the phase constants

The resultant motion of the system, which moves in the x-direction under the simultaneous effect of two harmonic oscillations can be found analytically.

Here we use the superposition principle to find out resultant displacement;

$$x = x_1 + x_2$$
$$x = a\cos(\omega t+\phi_1) + b\cos(\omega t+\phi_2).$$
$$x = (a\cos\phi_1 + b\cos\phi_2)\cos\omega t - (a\sin\phi_1 + b\sin\phi_2)\sin\omega t.$$

[using Trigonometry]

$$x = A\cos\delta\cos\omega t - A\sin\delta\sin\omega t.$$

$$\boxed{x = A\cos(\omega t + \delta)}$$

This implies that the resultant is also a SHM with same angular frequency 'ω'

Where, $a\cos\phi_1 + b\cos\phi_2 = A\cos\delta$

and $a\sin\phi_1 + b\sin\phi_2 = A\sin\delta$

Therefore, $A^2 = (a\cos\phi_1 + b\cos\phi_2)^2 + (a\sin\phi_1 + b\sin\phi_2)^2$

$$= a^2 + b^2 + 2ab\cos(\phi_2 - \phi_1).$$

and $$\tan\delta = \frac{a\sin\phi_1 + b\sin\phi_2}{a\cos\phi_1 + b\cos\phi_2}.$$

Here 'A' represents the amplitude of resultant SHM and 'δ' represents the phase constant.

Here, 'A' and 'δ', both parameters are constants i.e. time independent.

It is evident from the above equations, that the amplitude of the resulting oscillation is maximum when $\cos(\phi_2 - \phi_1) = +1$ or $(\phi_2 - \phi_1) = 2m\pi$; m is integer.

i.e. $\boxed{A_{max} = (a+b)}$

On the other hand, the resultant is minimum, when $\cos(\phi_2 - \phi_1) = -1$ or $(\phi_2 - \phi_1) = (2m+1)\pi$

i.e. $\boxed{A_{min} = (a-b)}$

(a) Superposition of two collinear Harmonic Oscillations having different frequencies (Beats).

Let us consider, two harmonic oscillations of different amplitudes 'a' and 'b' and different angular frequencies 'ω_1' and 'ω_2' respectively.

$$x_1 = a\cos\omega_1 t$$
$$x_2 = b\cos\omega_2 t$$

[For sake of simplicity, we have taken the phase constants of oscillations to be zero.]

Now, from superposition principle, the resultant will be,

$$x = x_1 + x_2 = a\cos\omega_1 t + b\cos\omega_2 t.$$

Let us now define, average frequency 'ω_a' and modulation frequency 'ω_m' as

$$\omega_a = \frac{1}{2}(\omega_1 + \omega_2) \quad \text{and} \quad \omega_m = \frac{1}{2}(\omega_2 - \omega_1) \quad \text{where } \omega_2 > \omega_1$$

So that,

$$\omega_1 = (\omega_a - \omega_m) \quad \text{and} \quad \omega_2 = (\omega_a + \omega_m).$$

Therefore,

$$x = a\cos(\omega_a - \omega_m)t + b\cos(\omega_a + \omega_m)t.$$

or,

$$x = (a+b)\cos\omega_m t\cos\omega_a t - (a-b)\sin\omega_a t\sin\omega_m t.$$

Let,

$$(a+b)\cos\omega_m t = A_m\cos\delta_m.$$
$$(a-b)\sin\omega_m t = A_m\sin\delta_m.$$

$$\therefore \boxed{x = A_m\cos(\omega_a t + \delta_m).}$$

$\Rightarrow$ This oscillation can be described as periodic with an angular frequency ω_a the average of two component frequencies.

Where

$$A_m^2 = a^2 + b^2 + 2ab\cos(2\omega_m t)$$
$$\delta_m = \tan^{-1}\frac{(a-b)\sin\omega_m t}{(a+b)\cos\omega_m t}.$$

The amplitude 'A_m' and phase constant 'δ_m' both vary with time, [which is different from previous case]

Beats :

When the frequencies of the SHMs are nearly equal, the resulting oscillation exhibits beats.

ie. $\omega_2 \simeq \omega_1$ so that, $\omega_m << \omega_a$.

The amplitude A_m of the resulting motion is maximum when

$$\cos(2\omega_m t) = +1$$

ie. $2\omega_m t = 0, 2\pi, 4\pi, \dots$

or, $(\omega_2 - \omega_1)t = 0, 2\pi, 4\pi, \dots$

or, $2\pi(\nu_2 - \nu_1)t = 0, 2\pi, 4\pi, \dots$ $\quad \left[\nu_1 = \frac{\omega_1}{2\pi}, \nu_2 = \frac{\omega_2}{2\pi}\right]$

or, $t = 0, \frac{1}{\nu_2 - \nu_1}, \frac{2}{\nu_2 - \nu_1}, \dots$

So, the time interval between two consecutive maxima is $\frac{1}{\nu_2 - \nu_1}$.

Similarly, the amplitude A_m of the resulting motion is minimum when

$$\cos(2\omega_m t) = -1$$

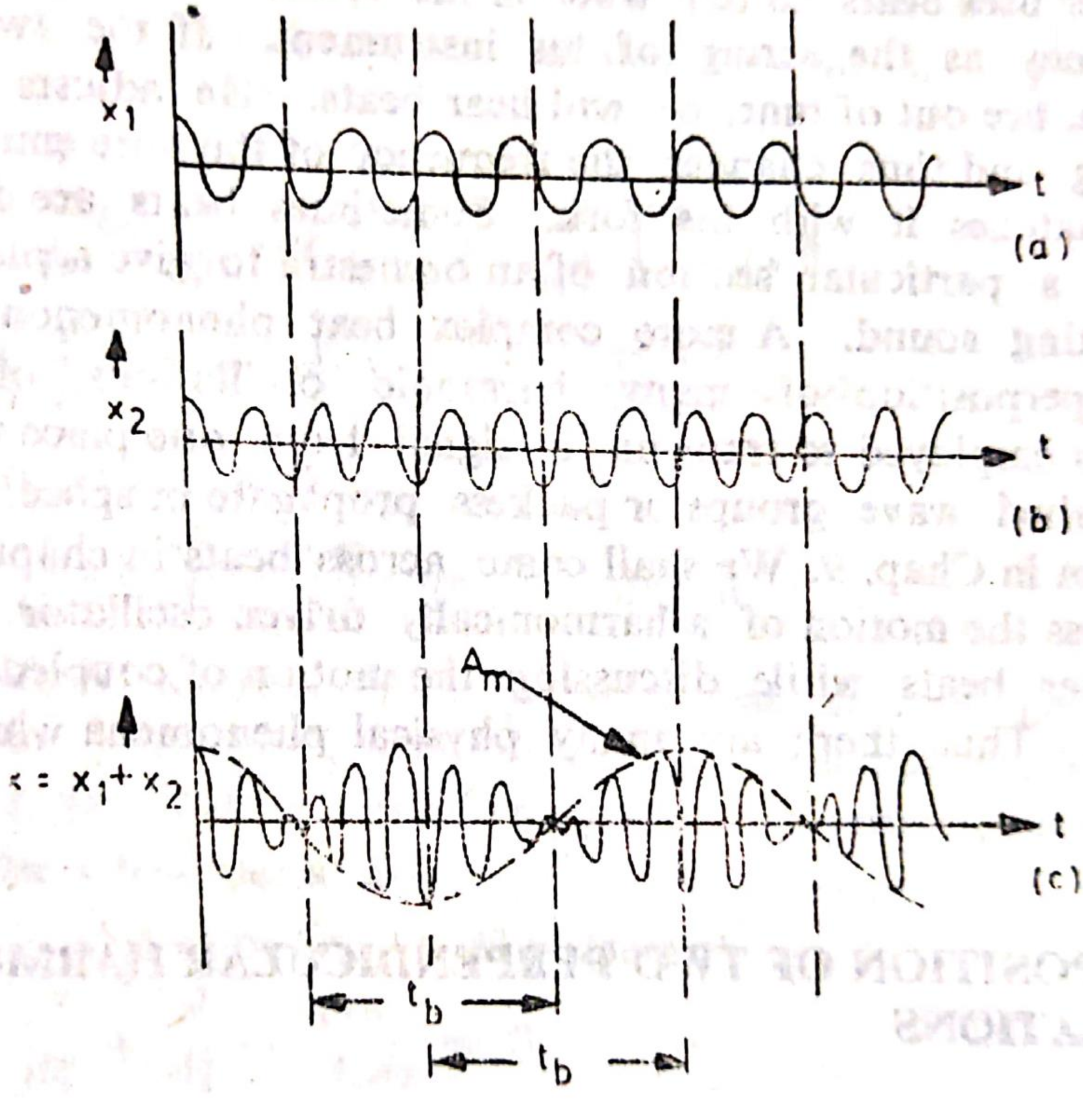
or, $2\omega_m t = \pi, 3\pi, 5\pi, \dots$

or, $t = \frac{1}{2(\nu_2 - \nu_1)}, \frac{3}{2(\nu_2 - \nu_1)}, \frac{5}{2(\nu_2 - \nu_1)}, \dots$ etc.

Here also the time interval between two consecutive minima is $\frac{1}{\nu_2 - \nu_1}$.

It is also noticable that, between any two maxima, there is a minima and vice-versa.

The periodic variation of amplitude of the motion, resulting from the superposition of slightly different frequency SHMs, is known as beats.



x1
t
(a)
x2
t
(b)
Am
x = x1 + x2
t
(c)
tb
tb

(b) Superposition of two perpendicular harmonic oscillations

Let, a particle moves under the simultaneous influence of two perpendicular harmonic oscillations of same frequency.

Two rectangular SHMs can be represented as,

$$x = a\cos\omega t$$

and $$y = b\cos(\omega t + \delta)$$

The two SHMs move along x-axis and y-axis respectively.
a & b be the amplitudes resp.
δ be their phase difference.

The resulting motion of the particle can be obtained as follows:

(i) Analytical method

$$y = b(\cos\omega t\cos\delta - \sin\omega t\sin\delta)$$

or, $$\frac{y}{b} = \frac{x}{a}\cos\delta - \sqrt{1-\frac{x^2}{a^2}}\sin\delta$$

using $x = a\cos\omega t$
i.e. $\frac{x}{a} = \cos\omega t$

or, $$\left(\frac{x}{a}\cos\delta - \frac{y}{b}\right) = \sqrt{1-\frac{x^2}{a^2}}\sin\delta.$$

Squaring both sides and on simplification we get.

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\delta = \sin^2\delta}$$

This is the general equation of an ellipse.
Hence, the path followed by a particle, which is subjected to two rectangular SHMs of equal frequencies is an ellipse.

Special cases:

(i) $\delta = 0$

In this case, $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0.$$

or, $$\left(y - \frac{b}{a}x\right)^2 = 0.$$

$$\therefore \boxed{y = \frac{b}{a}x}$$

This represents a pair of coincident straight lines passing through origin with positive slope.

(ii) $\delta = \pi/2$

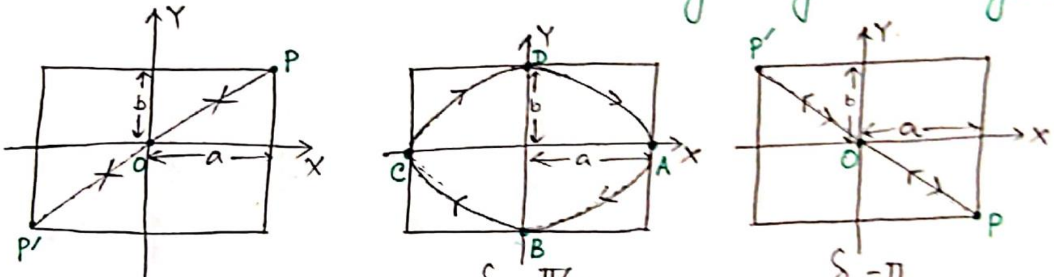
In this case, $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1}$ This represents equation of an ellipse whose principal axes lie along the x-axis and y-axis

(iii) $\delta = \pi$

In this case, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{2xy}{ab} = 0$

or, $\left(y + \dfrac{b}{a}x \right)^2 = 0$.

$\therefore \boxed{y = -\dfrac{b}{a}x}$ This represents a pair of coincident straight lines passing through origin with negative slope



$\delta = 0$ $\quad$ $\delta = \pi/2$ $\quad$ $\delta = \pi$

The directions of motion can be easily determined

Initially we have started with $x = a\cos\omega t$ and $y = b\cos(\omega t + \delta)$

Now by setting $\underline{\delta = 0}$, we will have,

$x = a\cos\omega t$ and $y = b\cos\omega t$.

at time $t = 0$; $x = a$, $y = b$ [At 'P']

at time $\omega t = \frac{\pi}{2}$; $x = 0$, $y = 0$ [At 'O']

at time $\omega t = \pi$; $x = -a$, $y = -b$ [At 'P'']

at time $\omega t = \frac{3\pi}{2}$; $x = 0$, $y = 0$. [At 'O']

at time $\omega t = 2\pi$; $x = a$, $y = b$. [At 'P']

The particle moves from 'P' to 'O' to 'P'' and traverse back to 'P' via 'O'. Thus the particle continues to vibrate along the straight line 'POP''.

This represents a <u>linearly polarized vibration</u>.

Next, by setting $\delta = \pi/2$

$x = a\cos\omega t$ and $y = b\cos(\omega t + \pi/2) = -b\sin\omega t$

At time $\omega t = 0$; $x = a$, $y = 0$ ['A']

At time $\omega t = \pi/2$; $x = 0$, $y = -b$ ['B']

At time $\omega t = \pi$; $x = -a$, $y = 0$ ['C']

At time $\omega t = \frac{3\pi}{2}$; $x = 0$, $y = b$ ['D']

At time $\omega t = 2\pi$; $x = a$, $y = 0$. ['A']

The particle traces out an ellipse in the clockwise sense. This is called the right-handed elliptically polarized vibration.

If $a = b$ ie. the amplitudes of the component vibrations are same, then the ellipse will turn out into a circle. $[x^2 + y^2 = a^2]$

Thus, two harmonic oscillations, at right angles to each other, of equal amplitudes and frequencies but with phase difference of $\pi/2$ are equivalent to uniform circular motion, the radius of the circle being equal to the amplitude of either oscillation.

Again by setting $\delta = \pi$

$x = a\cos\omega t$ and $y = b\cos(\omega t + \pi) = -b\cos\omega t$

At $t = 0$; $x = a$, $y = -b$ ['P']

At $\omega t = \pi/2$; $x = 0$, $y = 0$ ['O']

At $\omega t = \pi$; $x = -a$, $y = b$. ['P'']

At $\omega t = \frac{3\pi}{2}$; $x = 0$, $y = 0$ [O]

At $\omega t = 2\pi$; $x = a$, $y = -b$ ['P']

By setting $\delta = \frac{3\pi}{2}$

$x = a\cos\omega t$ and $y = b\cos(\omega t + \frac{3\pi}{2}) = b\sin\omega t$.

Which gives $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ $\Rightarrow$ equation of an ellipse.

But the motion is now in anti-clockwise sense.

This is called left-handed elliptically polarized vibration.

* you can check it by doing above mentioned process.

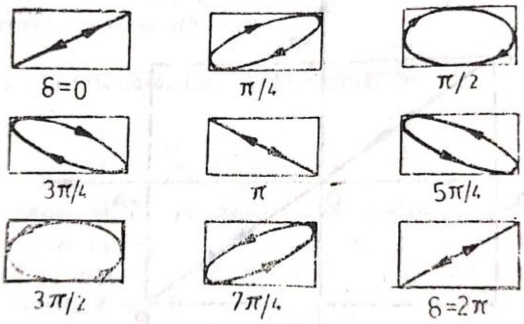


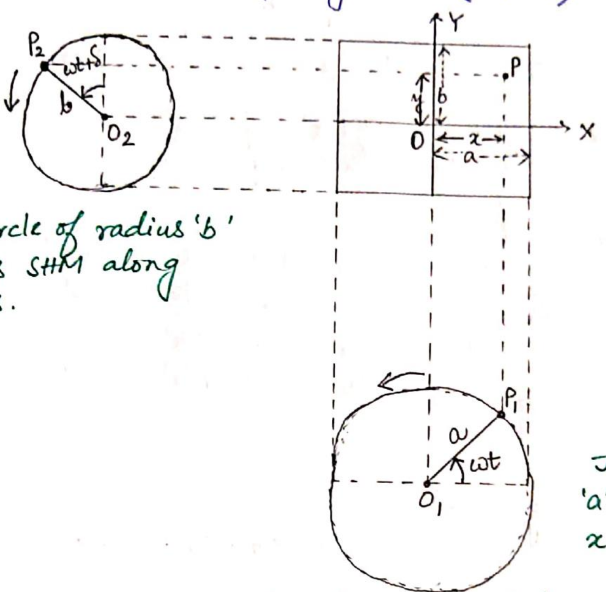
Fig. 2.8 Superposition of two perpendicular SHMs of the same frequency for various phase differences

(ii) Graphical method

The above results can also be realised graphically by a double application of rotating-vector technique.

We can start with our previous equations

$x = a\cos\omega t$ & $y = b\cos(\omega t+\delta)$



The circle of radius 'b' defines SHM along y-axis.

The circle of radius 'a' defines SHM along x-axis.

This is the geometrical representation of the superposition of two SHMs at right angle to each other.

O_1P_1 be the position of the rotating vector at time 't' (say). ie. the projection of O_1P_1 on x-axis gives the displacement at time 't'. O_2P_2 be the position of the rotating vector at 't'. Thus the projection of O_2P_2 on y-axis is the displacement at time 't'.

If the particle was subjected to both SHMs simultaneously, then its resultant displacement at time 't' would be OP.

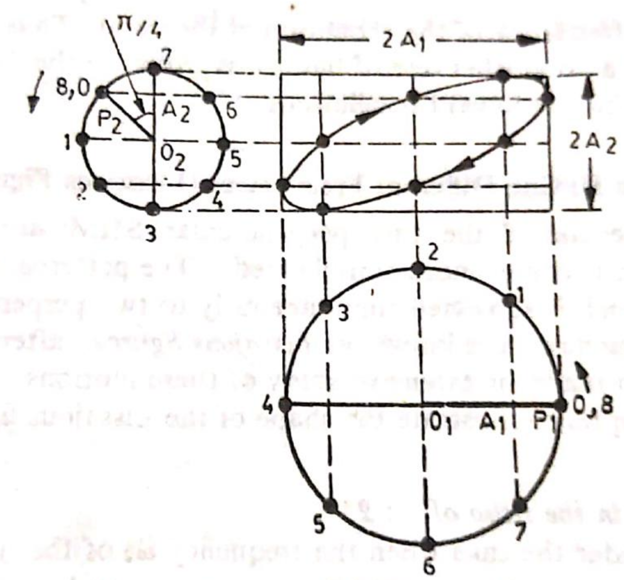


Fig. 2.11 Superposition of two perpendicular SMSs of the same frequency and a phase difference of $\pi/4$.

Oscillations having different frequencies.

When the frequencies of the two perpendicular SHMs are either equal or not equal, the patterns that are traced by a particle, are known as Lissajous figures.

Frequencies in the ratio 1 : 2.

Let, $x = a\cos\omega t$ and $y = b\cos(2\omega t + \delta)$.

The shape of the Lissajous figures can be obtained by both analytically and graphically.

Analytical method

$$\frac{y}{b} = \cos 2\omega t \cos\delta - \sin 2\omega t \sin\delta$$

$$= (2\cos^2\omega t - 1)\cos\delta - 2\sin\omega t\cos\omega t\sin\delta$$

$$= \left(2\frac{x^2}{a^2} - 1\right)\cos\delta - 2\frac{x}{a}\left(1 - \frac{x^2}{a^2}\right)^{1/2}\sin\delta$$

$$\left[\because \cos\omega t = \frac{x}{a}\right]$$

$$\therefore \left(\frac{y}{b} + \cos\delta\right) - \frac{2x^2}{a^2}\cos\delta = -\frac{2x}{a}\sqrt{1 - \frac{x^2}{a^2}}\sin\delta$$

Squaring bothsides and on simplification we get

$$\boxed{\left(\frac{y}{b} + \cos\delta\right)^2 + \frac{4x^2}{a^2}\left(\frac{x^2}{a^2} - 1 - \frac{y}{b}\cos\delta\right) = 0}.$$

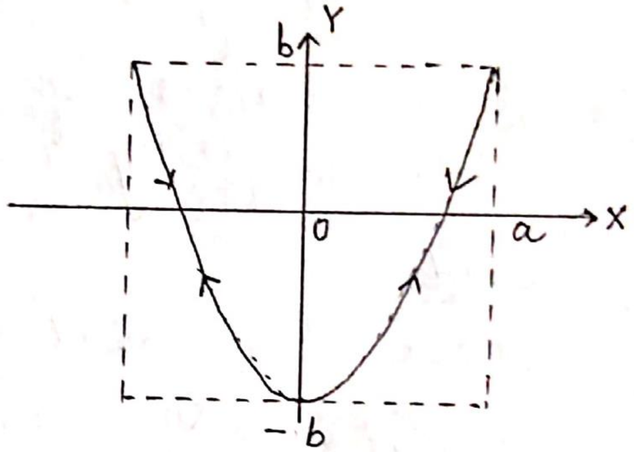
This is an equation of fourth degree, which in general represents a closed curve having two loops.

By setting $\delta = 0$, we get,

$$\left(\frac{y}{b} + 1 - \frac{2x^2}{a^2}\right)^2 = 0 .$$

or, $$\boxed{x^2 = \frac{a^2}{2b}(y+b)}$$

This represents two coincident parabolas with their vertices at $(0, -b)$.



The analytical method becomes very cumbersome for values of δ other than zero.
In such cases, the Lissajous figures can be realised conveniently by the graphical method.

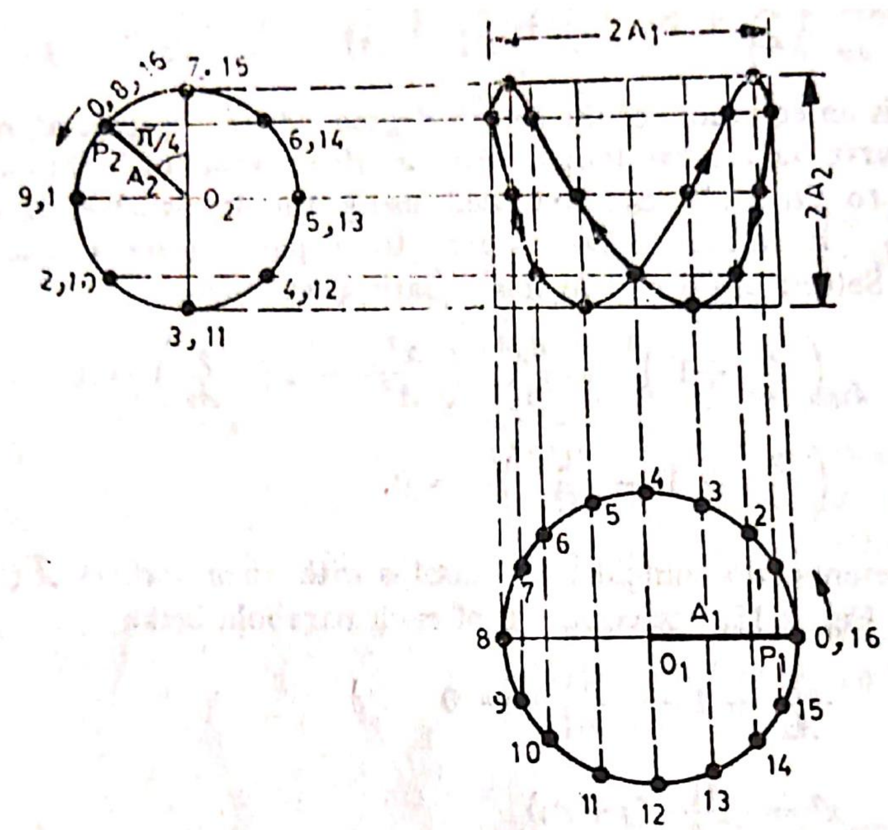


Fig. 2.13 Superposition of two perpendicular SHMs with frequencies in the ratio 1 : 2 and phase difference equal to π/4.

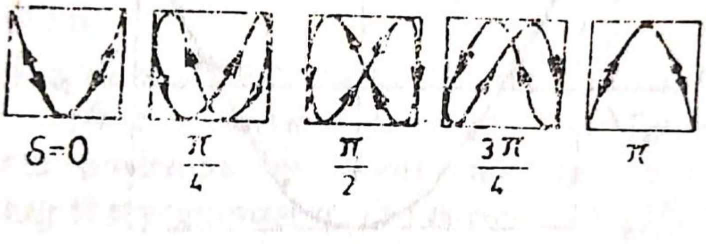


Fig. 2.14 Lissajous figures : $\omega_2 = 2\omega_1$ with various initial phase differences