VIVEKANANDA COLLEGE
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NAAC ACCREDITED ‘A’ GRADE

Topic: Lab frame and Center of mass frame.

Course Title: Quantum Scattering theory

Paper: Quantum Mechanics II

Unit: PHY-422

Semester: 2

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Name of the Department: PHYSICS

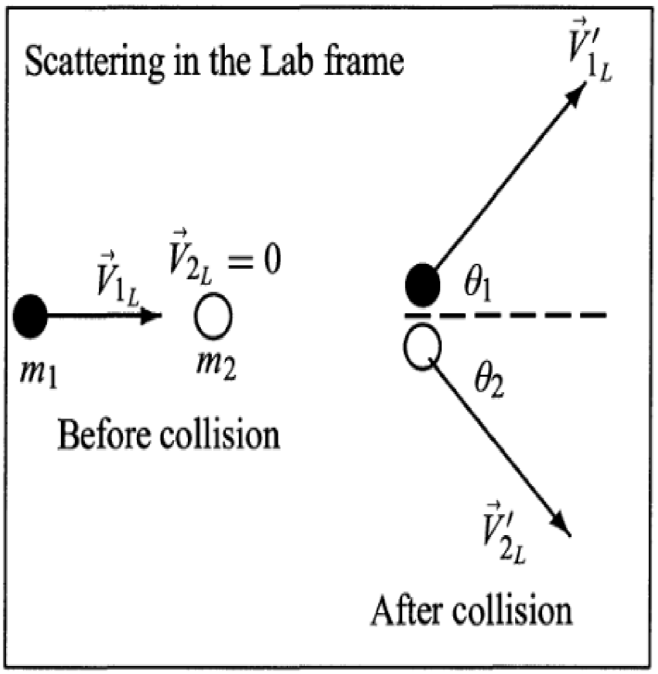
Connection between lab frame and Center of Mass frame:-

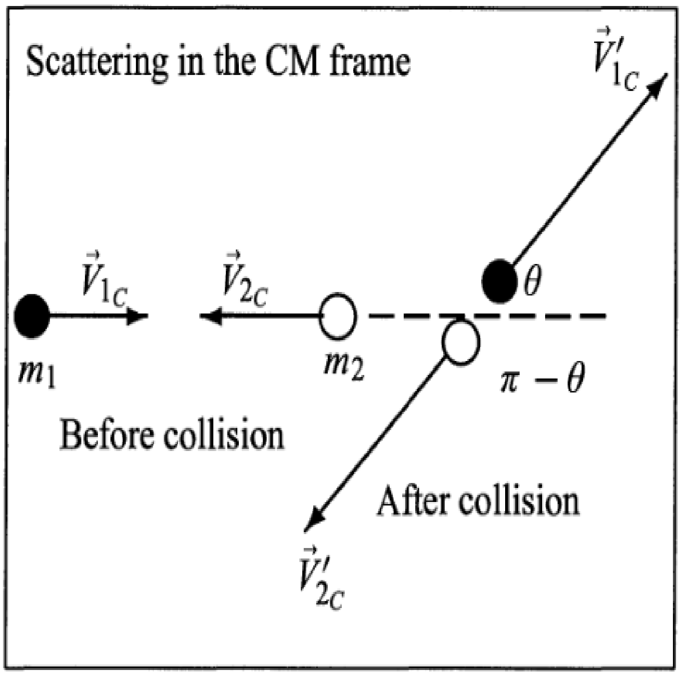
Most scattering experiments are carried out in the laboratory (Lab) frame in which the target is initially at rest while the projectiles are moving. Calculations of the cross sections are generally easier to perform within the center of mass (CM) frame in which the center of mass of the projectiles-target system is at rest (before and after collision).

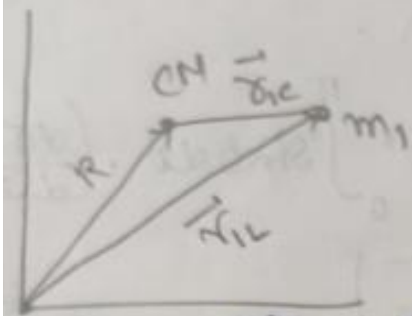
For Lab frame there are 6 d.o.f dut for CM frame only 3 d.o.f can be used. So calculation is easer in CM frame but observation must be done in Lab frame.

In order to be able to compare the experimental measurements with the theoretical calculations, one has to know how to transform the cross sections from one frame into the other. We should note that the total cross section σ is the same in both frames, since the total number of collisions that take place does not depend on the frame in which the observation is carried out. As for the differential cross sections $d\sigma(\theta,\phi)/d\Omega$ they are not the same in both frames, since the scattering angles (θ,ϕ) are frame dependent.

To find the connection between the Lab and CM cross sections, we need first to find how the scattering angles in one frame are related to their counterparts in the other. Let us consider the scattering of two (structure less, nonrelativistic) particles of masses m_1 and m_2: m_2 represents the target, which is initially at rest, and m_1 the projectile.







~~ott~~ Let $\vec{R}$ is the position vector of CM and $\vec{r}_{1c}$ is position of m_1 w.r.t CM and ~~R~~ $\vec{r}_{1L}$ is position of m_1 w.r.t origin (Lab frame)

$$\vec{r}_{1L} = \vec{R} + \vec{r}_{1c}$$

taking time derivative

$$\vec{v}_{1L} = \vec{v}_{CM} + \vec{v}_{1c} \quad \text{(before collision)}$$

& $$\vec{v}\,'_{1L} = \vec{v}\,'_{1c} + \vec{v}_{CM} \quad \text{(after collision)}$$

For elastic scattering — (from Geometry)

$$\vec{v}\,'_{1L}\cos\theta_1 = v'_{1c}\cos\theta + \vec{v}_{CM}$$

& $$\vec{v}\,'_{1L}\sin\theta_1 = \vec{v}\,'_{1c}\sin\theta$$

$$\therefore \tan\theta_1 = \frac{\vec{v}\,'_{1c}\sin\theta}{\vec{v}\,'_{1c}\cos\theta + \vec{v}_{CM}} = \frac{\sin\theta}{\cos\theta + \frac{\vec{v}_{CM}}{\vec{v}\,'_{1c}}}$$

We have $$\vec{V}_{CM} = \frac{m_1 \vec{v}_{1L} + m_2 v_{2L}}{m_1 + m_2} = \frac{m_1}{m_1 + m_2} \cdot \vec{v}_{1L}$$

as in Lab frame $\vec{v}_{2L} = 0$

$$\vec{V}_{1L} = \vec{V}_{1c} + \left(\frac{m_1}{m_1+m_2}\right)\cdot \vec{V}_{1L}$$

$$\therefore \vec{V}_{1c} = \vec{V}_{1L}\left(1 - \frac{m_1}{m_1+m_2}\right) = \frac{m_2}{m_1+m_2}\cdot \vec{V}_{1L}$$

From momentum conservation (before collision)

$$P_c = m_1 \vec{V}_{1c} - m_2 \vec{V}_{2c} = 0 \Rightarrow \boxed{\vec{V}_{2c} = \frac{m_1}{m_2}\cdot \vec{V}_{1c}}$$

after collision

$$P'_{1cx} = m_1 v'_{1c}\cos\theta + m_2 v'_{2c}\cos(\pi-\theta) = 0$$

$$\text{or } \boxed{\vec{V}'_{2c} = \frac{m_1}{m_2}\cdot \vec{V}'_{1c}}$$

For elastic scattering the total energy must conserved.

$$\frac{1}{2} m_1 v_{1c}^2 + \frac{1}{2} m_2 v_{2c}^2 = \frac{1}{2} m_1 v_{1c}'^2 + \frac{1}{2} m_2 v_{2c}'^2$$

$$\Rightarrow \frac{1}{2} m_1 v_{1c}^2 + \frac{1}{2} m_2 \left(\frac{m_1}{m_2}\right)^2 v_{1c}^2 = \frac{1}{2} m_1 v_{1c}'^2 + \frac{1}{2} m_2 \left(\frac{m_1}{m_2}\right)^2 v_{1c}'^2$$

$$\Rightarrow \frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right) v_{1c}^2 = \frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right) v_{1c}'^2$$

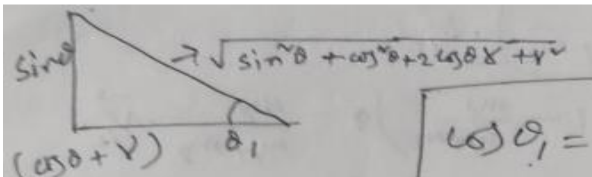
$$\therefore \boxed{\vec{V}_{1c} = \vec{V}'_{1c}} \quad \& \text{ Similarly } \boxed{\vec{V}_{2c} = \vec{V}'_{2c}}$$

$$\text{Then } \vec{V}'_{1c} = \vec{V}_{1L}\left(\frac{m_2}{m_1+m_2}\right)$$

Let us find out the ratio $\frac{V_{cM}}{V'_{1c}}$

$$\frac{V_{cM}}{V'_{1c}} = \frac{\left[m_1/(m_1+m_2)\right] V_{1L}}{\left[\frac{m_2}{m_1+m_2}\right] V_{1L}} = \frac{m_1}{m_2} = \gamma$$

$$\text{Then } \tan\theta_1 = \frac{\sin\theta}{\cos\theta + \left(\frac{V_{cM}}{V'_{1c}}\right)} = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}} = \frac{\sin\theta}{\cos\theta + \gamma}$$



$$\boxed{\cos\theta_1 = \frac{\cos\theta + \gamma}{\sqrt{1+2\gamma\cos\theta+\gamma^2}}}$$

Connection between Scattering Cross-section in CM and Lab Frame:-

The no. of scattered particles passing through an infinitesimal cross-section $d\sigma$ is same in both ~~&~~ frame.

$$d\sigma(\theta_1, \phi_1) = d\sigma(\theta, \phi)$$

They differs by solid angles in two frame

$$d\Omega_1 = \sin\theta_1\, d\theta_1\, d\phi_1 \quad \& \quad d\Omega = \sin\theta\, d\theta \cdot d\phi$$

Thus $\left(\frac{d\sigma_1}{d\Omega_1}\right)_{lab} \cdot d\Omega_1 = \left(\frac{d\sigma}{d\Omega}\right)_{CM} \cdot d\Omega$

or $\left(\frac{d\sigma_1}{d\Omega_1}\right)_{lab} = \left(\frac{d\sigma}{d\Omega}\right)_{CM} \cdot \frac{\sin\theta}{\sin\theta_1} \cdot \frac{d\theta}{d\theta_1} \cdot \frac{d\phi}{d\phi_1}$

~~do~~ due to azimuthally symmetric $\phi = \phi_1$

$$\therefore \left(\frac{d\sigma_1}{d\Omega_1}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{CM} \cdot \frac{d(\cos\theta)}{d(\cos\theta_1)}$$

$$\cos\theta_1 = \frac{\cos\theta + \gamma}{(1+\gamma^2+2\gamma\cos\theta)^{\frac{1}{2}}}$$

$$\therefore \frac{d(\cos\theta_1)}{d(\cos\theta)} = \frac{(1+0)}{(1+\gamma^2+2\gamma\cos\theta)^{\frac{1}{2}}} + \frac{(\cos\theta+\gamma)\left(-\frac{1}{2}\right)(2\gamma)}{(1+\gamma^2+2\gamma\cos\theta)^{3/2}}$$

$$= \frac{1+\gamma^2+2\gamma\cos\theta - \gamma(\cos\theta+\gamma)}{(1+\gamma^2+2\gamma\cos\theta)^{3/2}}$$

$$\frac{d(\cos\theta_1)}{d(\cos\theta)} = \frac{1+\gamma\cos\theta}{(1+\gamma^2+2\gamma\cos\theta)^{3/2}}$$

Finally $\left(\frac{d\sigma_1}{d\Omega_1}\right) = \left(\frac{d\sigma}{d\Omega}\right)\left[\frac{1+\gamma\cos\theta}{(1+\gamma^2+2\gamma\cos\theta)^{3/2}}\right]^{-1}$

$$\frac{d\sigma_1}{d\Omega_1} = \frac{d\sigma}{d\Omega}\cdot\left(\frac{(1+\gamma^2+2\gamma\cos\theta)^{3/2}}{1+\gamma\cos\theta}\right)$$

Now at $m_2 >> m_1$ then $\frac{m_1}{m_2} \to 0$ $\gamma \to 0$

and $\cos\theta_1 \to \cos\theta$

Then $\left(\frac{d\sigma_1}{d\Omega_1}\right) = \left(\frac{d\sigma}{d\Omega}\right)$

If $m_1 = m_2$ $\gamma = 1$

then $\cos\theta_1 = \frac{\cos\theta+1}{\sqrt{1+1+2\cos\theta}}$

and $\tan\theta_1 = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2(\theta/2)}$

$\tan\theta_1 = \tan(\theta/2)$

or $\boxed{\theta_1 = \theta/2}$

$$\left(\frac{d\sigma_1}{d\Omega_1}\right) = \left(\frac{d\sigma}{d\Omega}\right)\cdot\frac{(1+1+2\cos\theta)^{3/2}}{(1+\cos\theta)}$$

$$= \left(\frac{d\sigma}{d\Omega}\right)\cdot\frac{2^{3/2}(1+\cos\theta)^{3/2}}{(1+\cos\theta)}$$

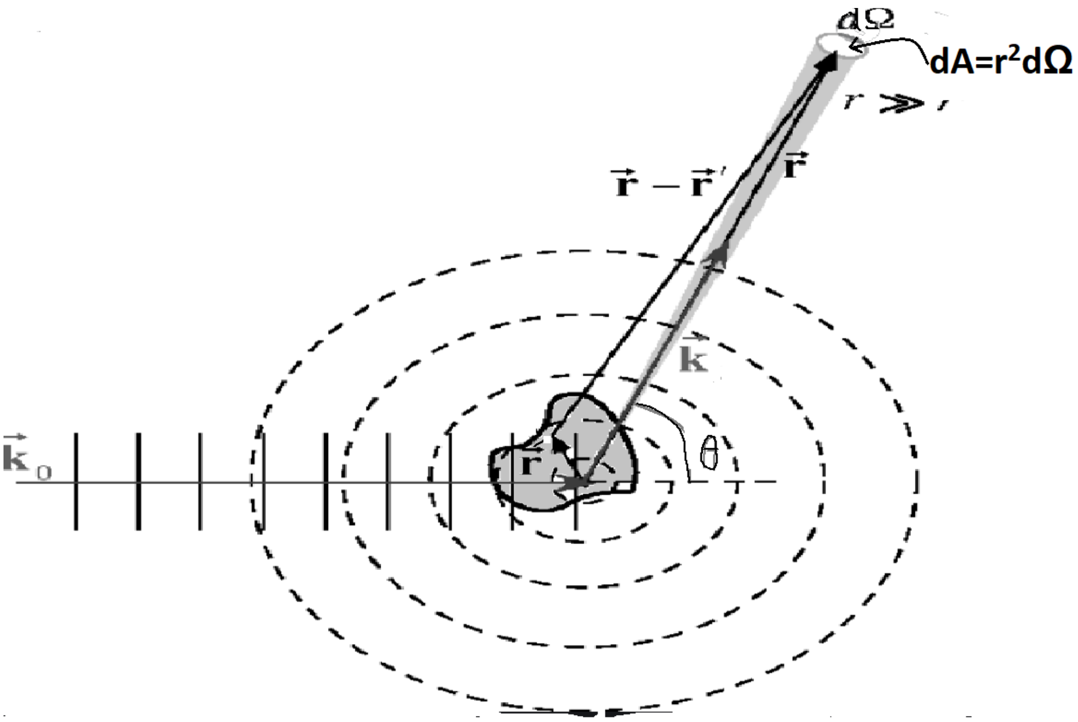
$$= \left(\frac{d\sigma}{d\Omega}\right)\cdot 2\sqrt{2}\,(1+\cos\theta)^{\frac{1}{2}}$$

$$= \left(\frac{d\sigma}{d\Omega}\right) 2\sqrt{2}\,\left(2\sin^2\theta/2\right)^{\frac{1}{2}}$$

$$\boxed{\left(\frac{d\sigma_1}{d\Omega_1}\right) = \left(\frac{d\sigma}{d\Omega}\right)\cdot 4\sin(\theta/2)}$$

for $m_1 = m_2$

Scattering amplitude:-



The wave function of two scatter may be represented as $\underline{\psi(\vec{r}_1,\vec{r}_2,t)} = \underline{\psi(\vec{r}_1,\vec{r}_2)\, e^{-iE_T t/\hbar}}$

$$\left[-\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\vec{r}_1,\vec{r}_2)\right]\psi(\vec{r}_1,\vec{r}_2) = E_T\,\psi(\vec{r}_1,\vec{r}_2)$$

$\vec{r} = |\vec{r}_1 - \vec{r}_2|$ then $V(\vec{r}_1,\vec{r}_2) = V(\vec{r})$

$\mu = \frac{m_1 m_2}{m_1 + m_2}$ Then the TISE be

$$\boxed{-\frac{\hbar^2}{2\mu}\nabla^2\psi(\vec{r}) + V(\vec{r})\,\psi(\vec{r}) = E_T\,\psi(\vec{r})}$$

when $r \le a$ the interaction will occur and when $r > a$ $V(r) = 0$.

<u>Incident particle</u> $r >> a$ then the SE be

$$(\nabla^2 + k_0^2)\,\psi_{in} = 0 \quad \text{where} \quad k_0^2 = \left(\frac{2\mu E_0}{\hbar^2}\right)$$

The solution $\psi_{in}(r) = A\, e^{i\vec{k}_0\cdot\vec{r}}$.

$\psi(r) = \psi_{in} + \psi_{sc}$. $\psi_{sc}(r) = A\, f(\theta,\phi)\,\frac{\psi_{in}}{r}$

$$\psi(r) \simeq A\left[e^{ik_0 z} + f(\theta,\phi)\,\frac{e^{i\vec{k}\cdot\vec{r}}}{r}\right]$$

$$\psi(r) = A\left[e^{ik_0 z} + f(\theta,\phi)\,\frac{e^{i\vec{k}\cdot\vec{r}}}{r}\right]$$

$f(\theta,\phi)$ is called <u>Scattering amplitude</u>.

Relation between f(θ,ϕ) and dσ (θ,ϕ) / dΩ :

$$\vec{j}_{in} = \frac{i\hbar}{2\mu}\left(\psi_{in}\vec{\nabla}\psi_{in}^{*} - \psi_{in}^{*}\vec{\nabla}\psi_{in}\right) \qquad \psi_{in} = A e^{ik_0 z}$$

$$= \frac{i\hbar}{2\mu}\left[A\, e^{ik_0 z}\cdot \vec{\nabla}A^{*}\left(e^{-ik_0 z}\right) - A^{*} e^{-ik_0 z}\vec{\nabla} A e^{ik_0 z}\right]$$

$$= \frac{i\hbar}{2\mu}A^2\left[e^{ik_0 z}(-ik_0)e^{-ik_0 z} - e^{-ik_0 z}(ik_0)e^{ik_0 z}\right]$$

$$= \frac{i\hbar}{2\mu}A^2(-2ik_0) = \frac{2k_0\hbar}{2\mu}\cdot A^2 = |A^2|\frac{\hbar k_0}{\mu}$$

Similarly we can write

$$\vec{j}_{sc} = \left|\frac{A f(\theta,\phi)}{r^2}\right|^2 \frac{\hbar k}{\mu}$$

Let $dN(\theta,\phi)$ is the no. of particles scattered into an element of solid angle $d\Omega$ in the direction (θ,ϕ) is passing through the area $r^2 d\Omega$ per unit time.

Then $dN(\theta,\phi)_{sc} = r^2 d\Omega \cdot \vec{j}_{sc}$

$$= |A|^2\frac{|f(\theta,\phi)|^2}{\cancel{r^2}}\left(\frac{\hbar k}{\mu}\right)\cdot \cancel{r^2}\, d\Omega$$

$$dN(\theta,\phi)_{in} = d\sigma\, \vec{j}_{in} = d\sigma\, |A^2|\,\frac{\hbar k_0}{\mu}$$

Then $d\sigma\, |A^2|\,\frac{\hbar k_0}{\mu} = d\Omega\, |A^2|\,|f(\theta,\phi)|^2\frac{k\hbar}{\mu}$

or $\boxed{\dfrac{d\sigma}{d\Omega} = \left(\dfrac{k}{k_0}\right)\cdot |f(\theta,\phi)|^2}$

For elastic scattering $k = k_0$ then

$$\left(\frac{d\sigma}{d\Omega}\right) = |f(\theta,\phi)|^2$$