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NAAC ACCREDITED ‘A’ GRADE

Topic Diffraction in a single slit (Fraunhofer diffraction)

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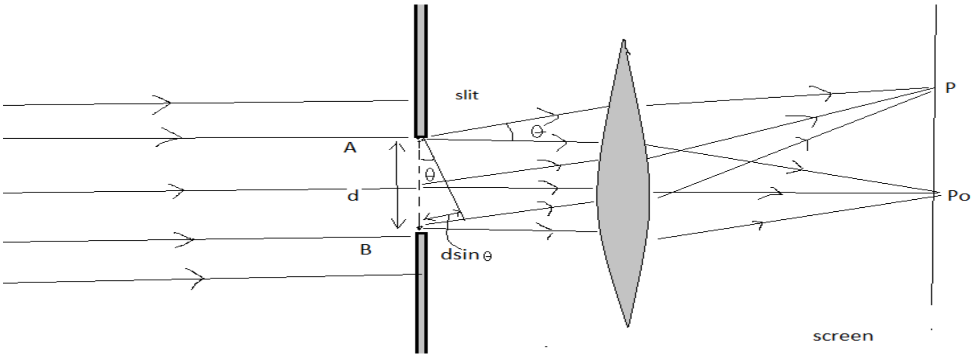
Diffraction in a single slit (Fraunhofer diffraction)

Superposition of waves: Waves from secondary sources have a form $y = ae^{i(wt+\varphi)}$

(Complex notation) where φ is the phase difference between the two adjacent secondary sources. 'w' is angular frequency of the source and as we use a monochromatic light source, 'w' is constant. The difference in phase part is only due to the difference between the path length of two superimposing waves.



Before going to geometrical configuration of path difference we have to go through the mathematical calculation.

Let there are N no. of sources between the points A & B and each source have a phase ϕ grater than the previous one.

The resultant disturbance can be written as –

$$Y = \sum_{j=1}^{n} y_j = ae^{i\omega t} + ae^{i(\omega t+\phi)} + ae^{i(\omega t+2\phi)} + \dots + ae^{i(\omega t+(N-1)\phi)}$$

$$\therefore y = ae^{i\omega t}\left[1 + e^{i\phi} + e^{2i\phi} + \dots + e^{i(N-1)\phi}\right]$$

the term in the parenthesis is a series sum of pure G.P. series. The 1st term is 1 & the common ratio is $e^{i\varphi}$. using the formula of sum of G.P series –

$$S = \text{1st term} \times \frac{1-(\text{Common ratio})^N}{1-\text{Common ratio}} \qquad N = \text{No of terms},\ i = \sqrt{-1}$$

$$\therefore y = ae^{i\omega t} \times \frac{1-e^{Ni\phi}}{1-e^{i\phi}}$$

$$= ae^{i\omega t} \times \left[\frac{1-e^{Ni\phi}}{1-e^{i\phi}}\right] \times \left[\frac{e^{iN\frac{\phi}{2}}}{e^{iN\phi/2}}\right]\left[\frac{e^{i\frac{\phi}{2}}}{e^{i\phi/2}}\right]$$

$$= ae^{i\omega t} \cdot \left[\frac{1-e^{Ni\phi}}{e^{iN\phi/2}}\right]\left[\frac{e^{i\phi/2}}{1-e^{i\phi}}\right]\left[\frac{e^{iN\frac{\phi}{2}}}{e^{i\phi/2}}\right]$$

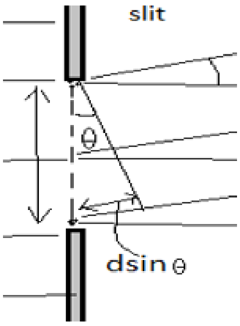
$$= ae^{i\omega t}\left[\frac{e^{-iN\phi/2} - e^{iN\frac{\phi}{2}}}{2i}\right]\left[\frac{1\times 2i}{e^{-i\frac{\phi}{2}} - e^{i\phi/2}}\right]\left[e^{i(N-1)\phi/2}\right]$$

$$= ae^{i\omega t} \cdot \frac{\sin\left(N\frac{\phi}{2}\right)}{\sin\left(\frac{\phi}{2}\right)} \cdot e^{i\left(\frac{N-1}{2}\right)\phi}$$

The last term $(N-1)/2\phi$ can be abbreviated as ϕ_{av} i.e. the average phase difference between N sources. $(N-1)\phi$ is the phase difference between extreme sources.

$$\therefore y = ae^{i(\omega t+\phi_{av})} \cdot \frac{\sin\left(\frac{N\phi}{2}\right)}{\sin\left(\phi/2\right)}$$

Let us comeback to the arrangement. The wave equation for the source point A can be written as $y = ae^{i(wt+\varphi 1)}$ and at B $y = ae^{i(wt+\varphi N)}$.



Path difference between the points A & B =dsin(θ).

So phase difference = dsin(θ)*(2π/λ)

$\phi_N - \phi_1$ = dsin(θ)*(2π/λ)

If we take the path difference between the two consecutive sources is d' then d=(N-1)d'

And corresponding phase difference between two consecutive sources is $\phi_{j+1} - \phi_j$ = δ=d'sin(θ)*(2π/λ).

So $\phi_N - \phi_1$ = (N-1)δ = (N-1)* d'sin(θ)*(2π/λ).

Nϕ=Nδ

$$y = a e^{i(\omega t + \phi_{av})} \cdot \frac{\sin\left(\frac{N\phi}{2}\right)}{\sin\left(\frac{\phi}{2}\right)}$$

$$= a e^{i(\omega t + \phi_{av})} \cdot \frac{\sin\left(\frac{N\delta}{2}\right)}{\sin\left(\delta/2\right)}$$

$$= a e^{i(\omega t + \phi_{av})} \cdot \frac{\sin\left(\frac{N}{2} \times \frac{2\pi}{\lambda} . d' \sin\theta\right)}{\sin\left(\frac{2\pi}{2\lambda} d' \sin\theta\right)}$$

$$y = a^{i(\omega t+\phi_{av})} \frac{\sin\left[\frac{N\pi}{\lambda}\cdot d' \sin\theta\right]}{\sin\left(\frac{\pi}{\lambda} d' \sin\theta\right)} \qquad d' = \frac{d}{N-1}$$

$$= a^{i(\omega t+\phi_{av})} \cdot \frac{\sin\left(\frac{N\pi}{\lambda}\cdot\frac{d}{(N-1)} \sin\theta\right]}{\sin\left[\frac{\cancel{N}\pi}{\lambda}\cdot\frac{d}{(N-1)} \sin\theta\right]}$$

$$= a^{i(\omega t+\phi_{av})} \frac{\sin\left(\frac{N}{N-1} \cdot \frac{d\pi}{\lambda}\sin\theta\right)}{\sin\left(\frac{1}{N-1}\cdot\frac{d\pi}{\lambda}\sin\theta\right)}$$

$$\left(\frac{d\pi}{\lambda}\cdot\sin\theta\right) \Rightarrow \alpha$$

then $y = a^{i(\omega t+\phi_{av})} \cdot \dfrac{\sin\left(\frac{N}{N-1}\alpha\right)}{\sin\left(\frac{1}{N-1}\cdot\alpha\right)}$

as $N \gg 1$ so $N-1 \approx N$ & $\frac{N}{N-1} \approx 1$

$$y = a^{i(\omega t+\phi_{av})} \cdot \frac{\sin\alpha}{\sin\left(\frac{\alpha}{N}\right)}$$

as N is very large $\frac{\alpha}{N}$ is very small

then $\sin\left(\frac{\alpha}{N}\right) \approx \left(\frac{\alpha}{N}\right)$

$$y = \frac{Na}{\alpha} e^{i(\omega t+\phi_{av})} \cdot \sin(\alpha)$$

$$y = Na e^{i(\omega t+\phi_{av})} \cdot \left[\frac{\sin\alpha}{\alpha}\right]$$

$A = Na\left[\dfrac{\sin\alpha}{\alpha}\right]$ and phase part $e^{i(\omega t+\phi_{av})}$

The intensity $I \propto A^2$

$$I = I_0 \cdot \frac{\sin^2\alpha}{\alpha^2} \quad \text{where } I_0 = (KN^2a^2)$$

We have to study the dependence of intensity I on α and through this α (which is d*π*sinθ/λ) the dependence of I on λ and θ.

Principal maxima:

I = $I_0 \frac{sin^2\alpha}{\alpha^2}$ has maximum value is I_0 as $\frac{sin^2\alpha}{\alpha^2}$ has maximum value=1 when α→ 0

$$\lim_{\alpha\to 0}\left(\frac{\sin\alpha}{\alpha}\right) = 1.$$

$$\alpha = 0: \quad \frac{d\pi\sin\theta}{\lambda} = 0: \theta = 0$$

So the principal maxima will occur at the position directly opposite to the slit. Hence maximum light will observed at that point.

Minimum intensity pattern: I = 0 if $\frac{\sin\alpha}{\alpha}$=0 or $\sin\alpha = 0$ but α≠ 0. It is possible if

α= $\pm m$π

or d*π*sinθ/λ = $\pm m$π

or dsinθ = mλ where m should not be zero as θ=0 position is conserved for principal maxima.

Secondary minima:-

$$I = I_0 \frac{\sin^2\alpha}{\alpha^2}$$

To find out the extrema $\frac{dI}{d\alpha} = 0$

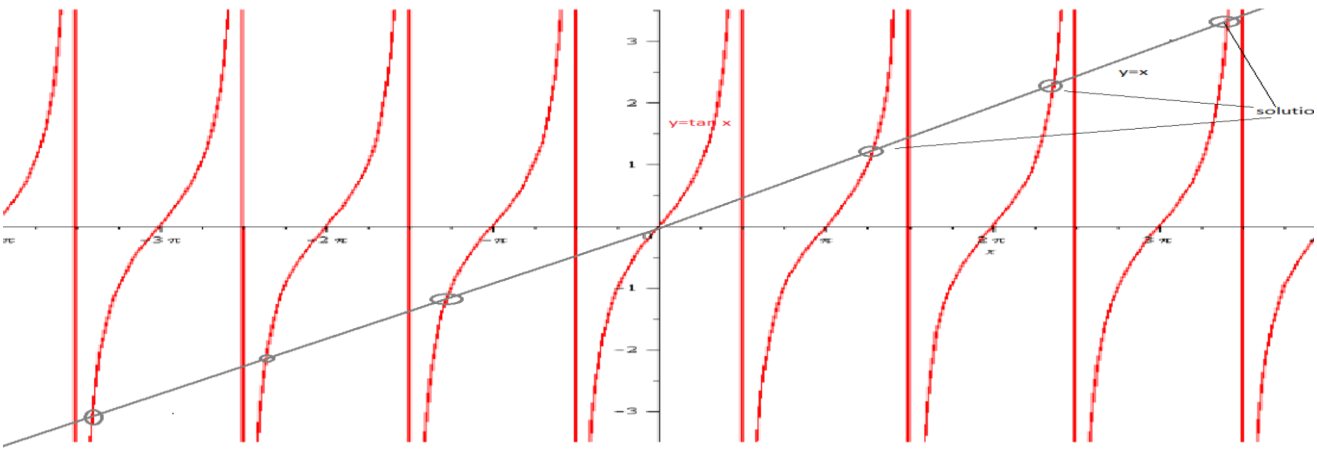
$$\frac{dI}{d\alpha} = \frac{d}{d\alpha}\left[I_0 \frac{\sin^2\alpha}{\alpha^2}\right]$$

$$= I_0 \left[\frac{(2\sin\alpha\cos\alpha)\alpha^2 - (\sin^2\alpha)\cdot 2\alpha}{\alpha^4}\right]$$

$$\therefore \frac{dI}{d\alpha} = 0 \text{ if } 2\alpha^2\sin\alpha\cos\alpha - \sin^2\alpha\cdot 2\alpha = 0$$

$$\cancel{\sin\alpha}\cos\alpha = \frac{\sin^{\cancel{2}}\alpha}{\alpha} \text{ or } \boxed{\tan\alpha = \alpha}$$

This is a transcendental equation. So to solve it we have to go through the graphical solution. By making the graph y = α & y = tan (α), and the cutting points indicate the solutions.



Order no.	Alfa from solution	Alfa from adjacent asymptote
1	4.49	4.71
2	7.725	7.85
3	10.904	10.99
4	14.066	14.137

The conditions for secondary maximas

$$\alpha = \pm(2m+1)\cdot\frac{\pi}{2}$$

$$\frac{d\pi}{\lambda}\sin\theta = \pm(2m+1)\frac{\pi}{2}$$

$$\boxed{d\sin\theta = \pm(2m+1)\frac{\lambda}{2}}$$

What are the intensity of the secondary maximas?

Intensity of Primary maxima = ~~I_0~~

$I_p = I_0$ (as $\frac{\sin^2\alpha}{\alpha^2} = 1$ at $\alpha = 0$)

1st secondary maxima have $\alpha = 4.49$

$\therefore I_1 = \frac{I_0}{4.49^2} = \frac{I_0}{22.21}$

$I_2 = \frac{I_0}{(7.725)^2} = \frac{I_0}{61.69}$ $\alpha = 7.725$

$I_3 = \frac{I_0}{(10.904)^2} = \frac{I_0}{120.9}$ $\alpha = 10.904$

Thus the 3rd maxima posses the intensity less than 1%.