VIVEKANANDA COLLEGE
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NAAC ACCREDITED ‘A’ GRADE

Topic: Effect of electric and magnetic field on plasma

Course Title: Introductory Plasma Physics

Paper: Classical Electrodynamics

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Name of the Department: Physics

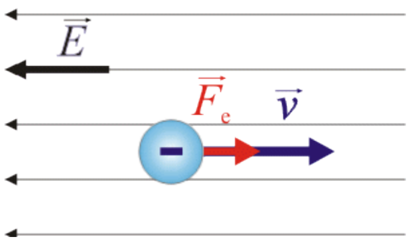
Introductory Plasma Physics

Motion of Charged Particles in Electric and Magnetic Field

The physical properties of plasma can be obtained by two methods: (i) the motion of individual particles composing the plasma such as electrons and positive ions are considered is called the particle description of the plasma (dilute plasma); (ii) plasma may be considered as two interacting charged fluids the negatively charged electron fluid and the positively charged ion fluid and the equation of magnetohydrodynamics apply (dense plasma).

❖ In an Electrostatic field:

Case I (*electric field and particle motion in* ***parallel***): If any charge particle with charge 'q' accelerated in the direction of the field, then;



$$m\frac{dv}{dt} = qE \Rightarrow \frac{dv}{dt} = \frac{q}{m}E \Rightarrow v = \frac{q}{m}Et + C$$

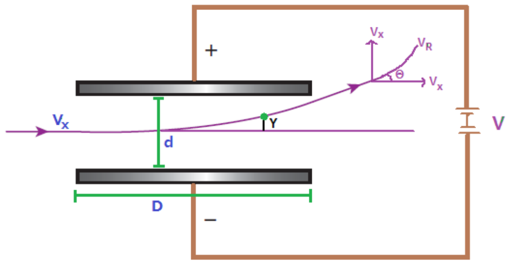
where C is a constant.

At $t = 0, v = 0$, so $C = 0$.

Hence, $v = \frac{q}{m}Et$. (10)

So, the particle moves in ***straight path*** along the direction or opposite direction of electric field depending upon the nature of charge of the particle.

Case II (*electric field and particle motion in* ***perpendicular*** *to each other*): Consider the electric filed along the y-axis and initial velocity of charge particle along x-axis. Due to Lorentz force, the charge particle accelerated along the electric filed (y-axis) with acceleration $\frac{qE}{m}$. So, from equation of motion we have, $y = 0.t + \frac{1}{2}\frac{qE}{m}t^2 = \frac{qE}{2m}t^2$.



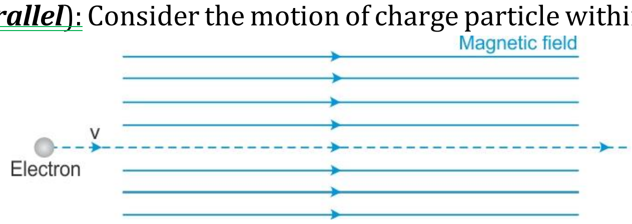
Also, along x-axis, $x = v_0 t$.

Eliminating t, $y = \frac{qE}{2mv_0^2}x^2$ (11)

which represent a ***parabolic*** equation.

❖ In a Uniform Magnetic field:

Case I (*magnetic field and particle motion in* ***parallel***): Consider the motion of charge particle within uniform magnetic field are in parallel. The force on the charge particles are zero, according to Lorentz force. Hence, the resulting path of charge particle must in ***straight*** line.



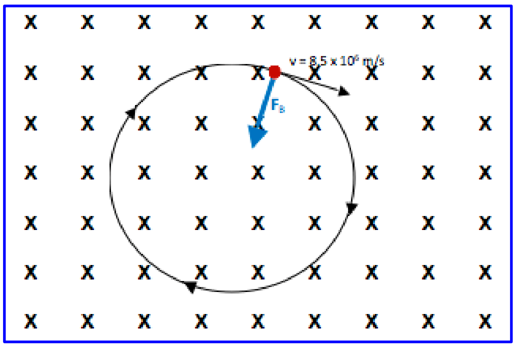
Case II (*magnetic field and particle motion in* ***perpendicular***): Consider the angle between the motion of charge particle and the magnetic force is $\pi/2$. Hence the force on charge particle is Hqv, along the direction perpendicular of both the magnetic field and direction of particle motion, act as the centripetal force on the charge particle and make the trajectory of ***circle***. Hence the equation of motion,

$$\frac{mv^2}{r} = Hqv \Rightarrow r_L = \frac{mv}{Hq}, \quad (12)$$

where r_L is the ***Larmor radius***.

In terms of energy conservation,

$$\frac{1}{2}mv^2 = W \Rightarrow mv = \sqrt{2mW}.$$

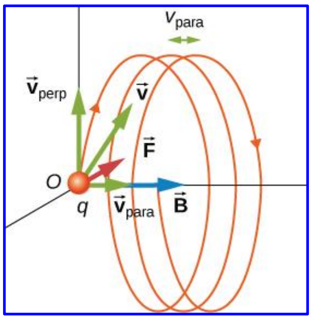


So, $r_L = \frac{\sqrt{2mW}}{HqZ}$;

for electron, $r_{Le} = \frac{3.4\sqrt{W}}{H}$ and

for proton, $r_{Lp} = \frac{143\sqrt{W}}{H}$.

And the corresponding ***Larmor frequency***, $\omega_H = \frac{qH}{m}$. (13)



Case III (***Any angle*** *between magnetic field and particle velocity*):

The perpendicular component of velocity response for ***circular*** motion and the parallel component response for the ***straight-line*** motion. As a result, the charge particle follows a ***helical path*** whose radius of gyration is $r_L = \frac{mv_\perp}{Hq}$ and the corresponding gyro-frequency is $\omega_H = \frac{qH}{m}$.

❖ **In presence of Uniform Electric and Magnetic field:**

Consider a charge particle moving within the region where the electric filed (E) lie in the $x - z$ plane and the magnetic field (H) along z-axis. The equation of motion of the charge particle under the action of both the electric and magnetic field be, $m\frac{d\vec{v}}{dt} = q[\vec{E} + \vec{v} \times \vec{H}]$

The z-component is $\frac{dv_z}{dt} = \frac{q}{m}E_z \Rightarrow v_z = \frac{qE_z}{m}t + v_{z0}$, which is straight forward acceleration along H.

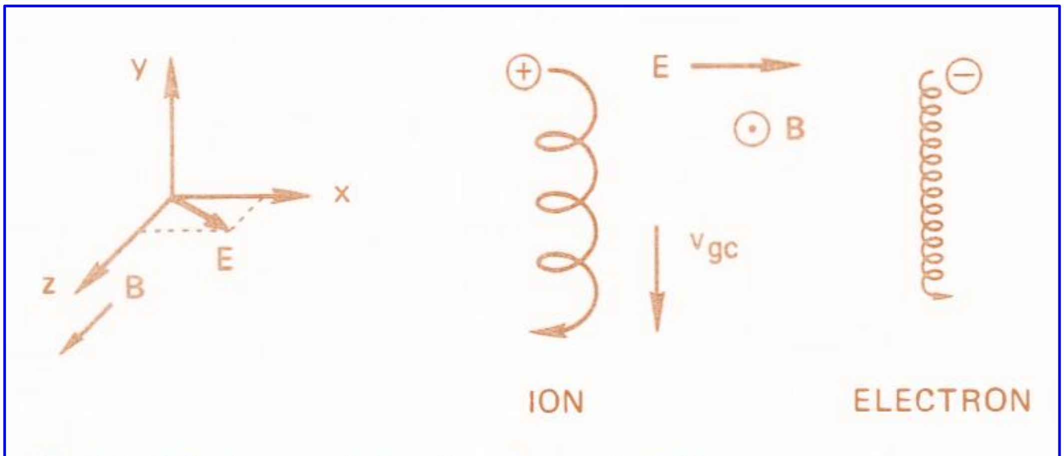
The transverse components are, $m\frac{dv_x}{dt} = q[E_x + v_y H]$ $\Rightarrow \frac{dv_x}{dt} = \frac{q}{m}E_x + \omega_H v_y$

where, $\omega_H = \frac{qH}{m}$, the electron cyclotron frequency.

Similarly, $\frac{dv_y}{dt} = -\omega_H v_x$

i.e., $\frac{d^2v_y}{dt^2} = -\omega_H \frac{dv_x}{dt} = -\omega_H\left[\frac{q}{m}E_x + \omega_H v_y\right]$

or, $\frac{d^2v_y}{dt^2} = -\omega_H^2\left[\frac{q}{H} + v_y\right]$



Particle drifts in crossed electric and magnetic fields.

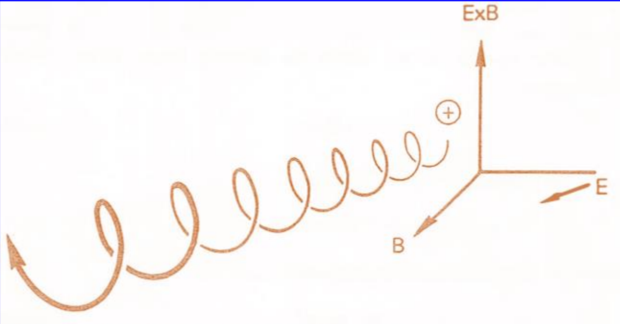
In absence of electric field,

$\frac{d^2 v_y}{dt^2} = -\omega_H^2 v_y$, which response for circular path motion.

In effect of electric field, v_y replaced by $v_y + \frac{E}{H}$, then, $v_x = v_\perp e^{i\omega_H t}$ and $v_y = v_\perp e^{i\omega_H t} - \frac{E}{H}$. (14)

So, the effective path of charge particle is a ***helical*** path with drift velocity, $v_d(cm/sec) = \frac{10^8\, E(volt/cm)}{H(Gauss)}$.

For particle of same velocity but different mass, the lighter one will have smaller gyration radius drift less per cycle. However, its gyration frequency is also larger, and the two effects exactly cancel. Two particles of the same mass but different energy would have the same ω_H. The slower one will have smaller r_L and hence gain less energy from E in a half-cycle. However, for less energetic particles the fractional change in r_L for a given change in energy is larger, and these two effects cancel. The three-dimensional orbit in space is therefore a slanted helix with changing pitch.



The actual orbit of a gyrating particle in space.

❖ Non-uniform Magnetic field:

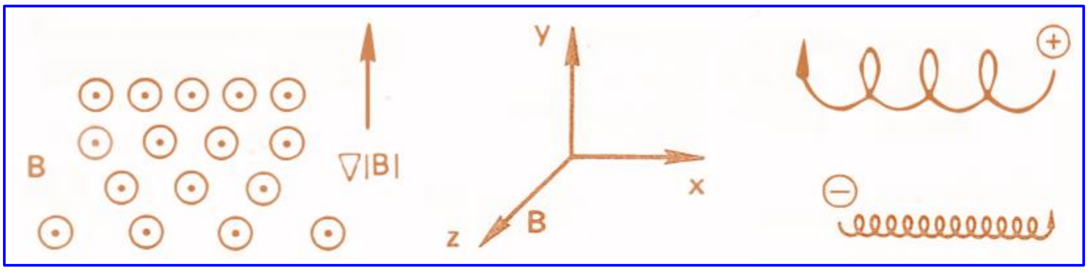
For uniform fields we were able to obtain exact expressions for the guiding centre drifts. As soon as we introduce inhomogeneity, the problem becomes too complicated to solve exactly. To get an approximate answer, it is customary to expand in the small ratio r_L/L, where L is the scale length of the inhomogeneity. This type of theory, called ***orbit theory***, can become extremely involved.

Case I ($\vec{\nabla}|H| \perp \vec{H}$; $Grad - H\ Drift$): Consider the magnetic lines of force are straight, but their density increases, say, in the y direction $\left[H_z = H_0 + y\frac{dH_z}{dy}\right]$. The gradient in $|\vec{H}|$ causes the Larmor radius to be larger at the bottom of the orbit than at the top, and this should lead to a drift, in opposite directions for ions and electrons, perpendicular to both $\vec{H}$ and $\vec{\nabla}|\vec{H}|$. The drift velocity should obviously be proportional to r_L/L and to $v_\perp$. The Lorentz force, $\vec{F} = q\vec{v}\times\vec{H}$, averaged over a gyration. Clearly, $F_x = 0$, since the particle spends as much time moving up as down. The force acting on the particle when it executes gyration in the magnetic field is,

$$F_y = -qHv_x = -qv_x\left[H_0 + y\frac{dH_z}{dy}\right] = -qv_\perp cos(\omega_H t)\left[H_0 \pm r_\perp cos(\omega_H t)\frac{dH_z}{dy}\right]$$

The time average of the 1st term is zero and that of the second term is ½.

Hence, $\left(F_y\right)_{av} = \pm\frac{1}{2}qv_\perp r_\perp\frac{dH_z}{dy}$. (15)



In this case the velocity of the guiding centre, $\boldsymbol{v_{\Delta H}} = \pm\frac{E}{H} = \pm\frac{qE}{qH} = \pm\frac{(F_y)_{av}}{qH} = \pm\frac{v_\perp r_\perp}{2H}\left(\frac{dH_Z}{dy}\right)$. (16)

Hence under this condition the electrons and ions will drift in the opposite directions and this will result in a charge separation and produce a strong electrostatic filed. This field in conjunction with the magnetic field will drift the plasma in a direction perpendicular to both the fields.

The equation (16) can be written in vector form as, $\vec{\boldsymbol{v}}_{\Delta H} = \pm\frac{v_\perp r_\perp}{2}\frac{\vec{H}\times\vec{\nabla}|\vec{H}|}{H^2}$. (17)

Note that the $\pm$ stands for the sign of the charge. The quantity $\vec{\boldsymbol{v}}_{\Delta H}$ is called the ***grad-H drift***; it is in opposite directions for ions and electrons and causes a current transverse to $\boldsymbol{H}$.

Case II (Curved H; *Curvature Drift*): Assume the magnetic lines of force to be curved with a constant radius of curvature R_c, and take $|\vec{H}|$ to be constant. Such a field does not obey Maxwell's equations in a vacuum, so in practice the grad-$|\vec{H}|$ drift will always be added to the effect derived here. A guiding centre drift arises from the centrifugal force felt by the particles as they move along the field lines in their thermal motion. If $v_\parallel^2$ denotes the average square of the component of random velocity along $\vec{H}$, the average centrifugal force is $\vec{\boldsymbol{F}}_{cf} = \frac{mv_\parallel^2}{R_c}\hat{\boldsymbol{r}} = mv_\parallel^2\frac{\vec{R}_c}{R_c^2}$. (18)

This gives the curvature-drift, $\vec{\boldsymbol{v}}_R = \frac{1}{q}\frac{\vec{F}_{cf}\times\vec{H}}{|\vec{H}|^2} = \frac{mv_\parallel^2}{q|\vec{H}|^2}\frac{\vec{R}_c\times\vec{H}}{R_c^2}$ (19)

Now compute the ***grad-H*** drift which accompanies this when the decrease of $|\vec{H}|$ with radius is taken into account. In a vacuum, $\vec{\nabla}\times\vec{H} = 0$. In the cylindrical coordinates, $\vec{\nabla}\times\vec{H}$ has only a z component, since $\vec{H}$ has only a θ component and $\vec{\nabla}|\vec{H}|$ only an $\vec{r}$ component. Then,

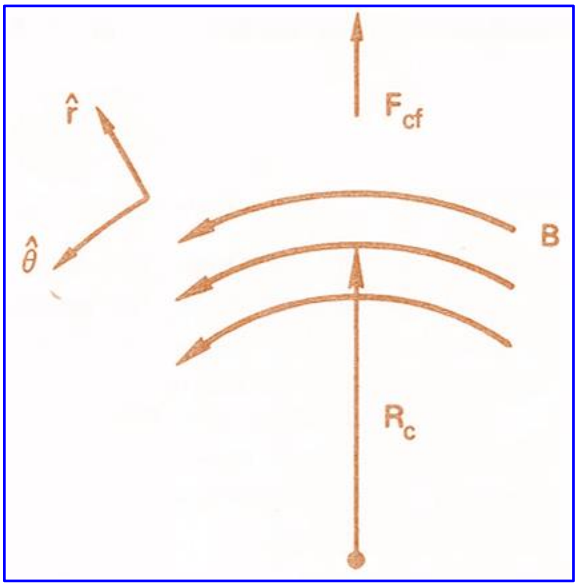
$$\left(\vec{\nabla}\times\vec{H}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta) = 0 \quad \text{i.e.,} \quad H_\theta \propto \frac{1}{r} \tag{20}$$

$$\text{Thus,} \quad |\vec{H}| \propto \frac{1}{R_c} \quad \text{and} \quad \frac{\vec{\nabla}|\vec{H}|}{|\vec{H}|} = -\frac{\vec{R}_c}{R_c^2} \tag{21}$$

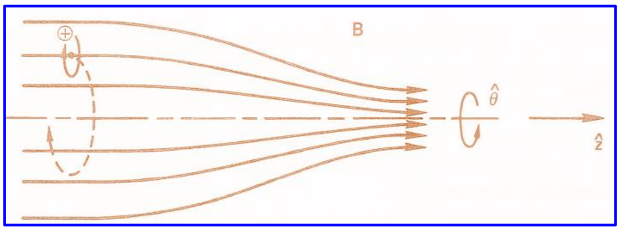
From equation (17), $$\vec{\boldsymbol{v}}_{\Delta H} = \pm\frac{v_\perp r_\perp}{2|\vec{H}|^2}\vec{H}\times|\vec{H}|\frac{\vec{R}_c}{R_c^2} = \pm\frac{1}{2}\frac{v_\perp^2}{\omega_c}\frac{\vec{R}_c\times\vec{H}}{R_c^2\,|\vec{H}|} = \frac{1}{2}\frac{mv_\perp^2}{q}\frac{\vec{R}_c\times\vec{H}}{R_c^2\,|\vec{H}|^2} \tag{22}$$

The total drift in a curved vacuum field: $$\vec{\boldsymbol{v}}_R + \vec{\boldsymbol{v}}_{\Delta H} = \frac{m}{q}\frac{\vec{R}_c\times\vec{H}}{R_c^2\,|\vec{H}|^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right) \tag{23}$$

It is unfortunate that these drifts add. This means that if one bends a magnetic field into a torus for the purpose of confining a thermonuclear plasma, the particles will drift out of the torus no matter how one juggles the temperatures and magnetic fields.



Case III ($\vec{\nabla}|H| \parallel \vec{H}$; Magnetic Mirrors): Consider a magnetic field which is pointed primarily in the $\boldsymbol{z}$ direction and whose magnitude vary in the z direction. Let the field be axisymmetric, with $H_\theta = 0$ and $\partial/\partial\theta = 0$. Since the magnetic lines of force converge and diverge, there is necessarily a component H_r. We can see that this gives rise to a force which can trap a particle within a magnetic field.



The H_r can be obtained by using relation $\vec{\nabla}.\vec{H} = 0$: $$\frac{1}{r}\frac{\partial}{\partial r}(rH_r) + \frac{\partial H_z}{\partial z} = 0 \qquad (24)$$

If $\frac{\partial H}{\partial z}$ is given at $\boldsymbol{r} = \boldsymbol{0}$ and does not vary much with $\boldsymbol{r}$, then

$$rH_r = -\int_0^r r\frac{\partial H_z}{\partial z}\,dr \simeq -\frac{r^2}{2}\left[\frac{\partial H_z}{\partial z}\right]_{r=0} \qquad \text{or,} \qquad H_r = -\frac{r}{2}\left[\frac{\partial H_z}{\partial z}\right]_{r=0} \qquad (25)$$

The variation of $|\vec{H}|$ with r causes a grad-H drift of guiding centres about the axis of symmetry, but there is no radial grad-H drift, because $\partial H/\partial\theta = 0$. The components of the Lorentz force are

$$F_r = q(v_\theta H_z - v_z H_\theta); \quad F_\theta = q(-v_r H_z + v_z H_r); \quad F_z = q(v_r H_\theta - v_\theta H_r) \qquad (26)$$

Two term of above three equation vanish if $H_\theta = 0$, and 1st terms of F_r and F_θ gives rise to the usual Larmor gyration. The 2nd term of F_θ vanishes on the axis; when it does not vanish, this azimuthal force causes a drift in the radial direction. This drift merely makes the guiding centres follow the lines of force.

Hence, $$F_z = \frac{1}{2}qv_\theta r(\partial H_z/\partial z). \qquad (27)$$

For simplicity, consider a particle whose guiding centre lies on the axis. Then v_θ is a constant during a gyration; depending on the sign of q, v_θ is $\mp v_\perp$. Since $r = r_L$, the average force is

$$(F_z)_{av} = \mp\frac{1}{2}qv_\perp r_L\frac{\partial H_z}{\partial z} = \mp\frac{1}{2}q\frac{v_\perp^2}{\omega_c}\frac{\partial H_z}{\partial z} = -\frac{1}{2}\frac{mv_\perp^2}{H}\frac{\partial H_z}{\partial z} \qquad (28)$$

The magnetic moment of the gyrating particle to be $$\mu = \frac{1}{2}mv_\perp^2/H \qquad (29)$$

Hence, $$(F_z)_{av} = -\mu\frac{\partial H_z}{\partial z} \qquad (30)$$

This is a specific example of the force on a diamagnetic particle, which in general can be written, $\boldsymbol{F}_\parallel = -\mu\frac{\partial H}{\partial s} = -\mu\nabla_\parallel H$, where $\boldsymbol{ds}$ is a line element along $\vec{H}$. Note that the definition [29] is the same as the usual definition for the magnetic moment of a current loop with area A and current $\boldsymbol{I}$: $\mu = IA$. In the case of a singly charged ion, $\boldsymbol{I}$ is generated by a charge $\boldsymbol{q}$ coming around $\omega_c/2\pi$ times a second: $I = q\omega_c/2\pi$. The area $\boldsymbol{A}$ is $\pi r_L^2 = \pi v_\perp^2/\omega_c^2$. Thus $\mu = \frac{\pi v_\perp^2}{\omega_c^2}\frac{q\omega_c}{2\pi} = \frac{1}{2}\frac{v_\perp^2}{\omega_c} = \frac{mv_\perp^2}{2H}$. As the particle moves into regions of stronger or weaker $\boldsymbol{H}$, its Larmor radius changes, but μ remains invariant. To prove this, consider the component of the equation of motion along $\boldsymbol{H}$: $$m\frac{dv_\parallel}{dt} = -\mu\frac{\partial H}{\partial s} \qquad (31)$$

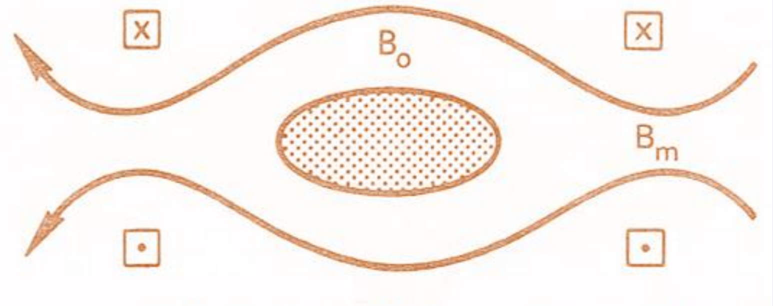
Multiplying by $v_\parallel$ on the left and its equivalent ds/dt on the right,

$$mv_\parallel\frac{dv_\parallel}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu\frac{\partial H}{\partial s}\frac{ds}{dt} = -\mu\frac{dH}{dt} \qquad (32)$$

Here $\frac{dH}{dt}$ is the variation of $\boldsymbol{H}$ as seen by the particle; $\boldsymbol{H}$ itself is constant. The particle's energy must be conserved, so we have $$\frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right) = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2 + \mu H\right) = 0 \qquad (33)$$

With equation (32), this becomes $$-\mu\frac{dH}{dt} + \frac{d}{dt}(\mu H), \qquad \text{i.e.,} \qquad \frac{d\mu}{dt} = 0 \qquad (34)$$

The invariance of $\boldsymbol{\mu}$ is the basis for one of the primary schemes for plasma confinement: the *magnetic mirror*. As a particle moves from a weak-field region to a strong-field region in the course of its thermal motion, it sees an increasing $\boldsymbol{H}$, and therefore its $\boldsymbol{v}_\perp$ must increase in order to keep $\boldsymbol{\mu}$ constant. Since its total energy must remain constant, $\boldsymbol{v}_\parallel$ must necessarily decrease. If $\boldsymbol{H}$ is high enough in the "throat" of the mirror, $\boldsymbol{v}_\parallel$ eventually becomes zero; and the particle is "reflected" back to the weak-field region. It is, of course, the force $\boldsymbol{F}_\parallel$ which causes the reflection. The nonuniform field of a simple pair of coils forms two magnetic mirrors between which a plasma can be trapped. This effect works on both ions and electrons.



A plasma trapped between magnetic mirrors.

❖ Time varying Magnetic field:

Now consider the magnetic field varies with time. Since the Lorentz force is always perpendicular to $\boldsymbol{v}$, a magnetic field itself cannot impart energy to a charged particle. However, associated with $\boldsymbol{H}$ is an electric field given by

$$\vec{\nabla} \times \vec{E} = -\dot{\vec{H}} \tag{35}$$

and this can accelerate the particles. We can no longer assume the fields to be completely uniform. Let $\vec{v}_\perp = \frac{d\vec{l}}{dt}$ be the transverse velocity $\vec{l}$ being the element of path along a particle trajectory (with $\vec{v}_\parallel$ neglected). Taking the scalar product of the equation of motion with $\vec{v}_\perp$,

$$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = q\vec{E}.\vec{v}_\perp = q\vec{E}.\frac{d\vec{l}}{dt} \tag{36}$$

The change in one gyration is obtained by integrating over one period:

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = \int_0^{2\pi/\omega_c} q\vec{E}.\frac{d\vec{l}}{dt}dt$$

If the field changes slowly, we can replace the time integral by a line integral over the unperturbed orbit:

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = \oint q\vec{E}.d\vec{l} = q\int_s (\vec{\nabla}\times\vec{E}).d\vec{S} = -q\int_s \dot{\vec{H}}.d\vec{S} \tag{37}$$

Here $\vec{S}$ is the surface enclosed by the Larmor orbit and has a direction given by the right-hand rule when the fingers point in the direction of $\vec{v}$. Since the plasma is diamagnetic, we have $\dot{\vec{H}}.d\vec{S} < 0$ for ions and > 0 for electrons. Then Eq. [37] becomes $\delta\left(\frac{1}{2}mv_\perp^2\right) = \pm q\dot{\vec{H}}\pi r_L^2 = \pm q\pi\dot{\vec{H}}\frac{v_\perp^2}{\omega_c}.\frac{m}{\pm q\,|\vec{H}|} = \frac{\frac{1}{2}mv_\perp^2}{|\vec{H}|}.\frac{2\pi\dot{\vec{H}}}{\omega_c}$ (38)

The quantity $\frac{2\pi\dot{\vec{H}}}{\omega_c} = \frac{\dot{\vec{H}}}{f_c}$, is just the change $\boldsymbol{\delta}\vec{H}$ during one period of gyration. Thus $\delta\left(\frac{1}{2}mv_\perp^2\right) = \mu\delta\vec{H}$ (39)

Since the left-hand side is $\delta(\mu\delta\vec{H})$, we have the desired result $\delta\mu = 0$ (40)

The magnetic moment is invariant in slowly varying magnetic fields.

As the $\vec{H}$ field varies in strength, the Larmor orbits expand and contract, and the particles lose and gain transverse energy. This exchange of energy between the particles and the field is described very sim ply by Eq. [40]. The invariance of $\boldsymbol{\mu}$ allows us to prove easily the following well-known theorem:

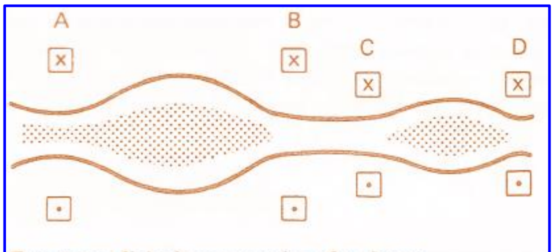
The magnetic flux through a Larmor orbit is constant.

The flux $\boldsymbol{\Phi}$ is given by $\boldsymbol{HS}$, with $\boldsymbol{S} = \boldsymbol{\pi r_L^2}$. Thus

$$\boldsymbol{\Phi} = \boldsymbol{H\pi}\frac{\boldsymbol{v_\perp^2}}{\boldsymbol{\omega_c^2}} = \boldsymbol{H\pi}\frac{\boldsymbol{v_\perp^2 m^2}}{\boldsymbol{q^2H^2}} = \frac{\boldsymbol{2\pi m}}{\boldsymbol{q^2}}.\frac{\frac{1}{2}\boldsymbol{mv_\perp^2}}{\boldsymbol{H}} = \frac{\boldsymbol{2\pi m}}{\boldsymbol{q^2}}.\boldsymbol{\mu} \qquad (41)$$

Therefore, Φ is constant if μ is constant.

This property is used in a method of plasma heating known as ***adiabatic compression***. Figure shows a schematic of how this is done. A plasma is injected into the region between the mirrors A and B. Coils A and B are t hen pulsed to increase $\boldsymbol{H}$ and hence $v_\perp^2$. The heated plasma can then be transferred to the region C-D by a further pulse in A, increasing the mirror ratio there. The coils C and D are t hen pulsed to further compress and heat the plasma. Early magnetic mirror fusion devices employed this type of heating. Adiabatic compression has also been used successfully on toroidal plasmas and is an essential element of laser-driven fusion schemes using either *magnetic* or *inertial confinement*.



Two-stage adiabatic compression of a plasma.

The above property used for plasma heating by ***magnetic pumping***. The magnetic pumping consists in periodic magnetic compression and decompressions of the plasma. This must take place in a time interval very large compare to the Larmor period, but at the same time, very small compared to the relaxation time necessary for the achievement of thermal equilibrium.